FALL 2007

SEMINAR ON PRESUPPOSITION

EXCURSUS ON TRIVALENCE:

SUPERVALUATIONS VS. STRONG KLEENE

I. Strong Kleene

(1) **Strong Kleene Logic**

a. $I_s(\neg F) = 1$ iff $I_s(F) = 0$; $= 0$ iff $I_s(F) = 1$
b. $I_s(F \wedge G) = 1$ iff $I_s(F) = I_s(G) = 1$; $= 0$ iff $I_s(F) = 0$ or $I_s(G) = 0$
c. $I_s(\forall x\ F) = 1$ iff for each d in the domain, $I_{s[x\rightarrow d]}(F) = 1$; $= 0$ iff for some d in the domain, $I_{s[x\rightarrow d]}(F)=0$.

Other connectives are defined using the *usual* interdefinability rules.

II. Supervaluations

(2) **A simple supervaluationist logic**

• Definition 1

I' is a bivalent extension of I just in case for every proposition p of type t, for every predicate P of type <e, t>, and for every individual of type d:
if $I(p) \neq u$, $I'(p) = I(p)$; otherwise, $I'(p) \in \{0, 1\}$
if $[I(P)](d) \neq u$, $[I'(P)](d) = [I(P)](d)$; otherwise, $[I'(P)](d) \in \{0, 1\}$

• Definition 2

Let F be a formula, interpreted with respect with respect to a valuation i (which assigns truth values among {0, 1, u} to atomic elements). Then i is extended to an interpretation I for the entire language by using the following rules, together with a *classical* semantics for bivalent interpretations:

$I(F) = 1$ iff for every bivalent extension I' of i, $I'(F) = 1$
$I(F) = 0$ iff for every bivalent extension I' of i, $I'(F) = 0$
$I(F) = \#$ otherwise.

(3) **Examples**

Example 1: $i(p) = u$
$I(p) = \#$ because there is a bivalent extension I' of i for which $I'(p) = 1$ and another bivalent extension I" of i for which $I''(p) = 0$.

Example 2: i(p) = u; i(q) = 1
I(p ∧ q) = # because there is a bivalent extension I' of i for which I'(p) = I'(q) = 1 - hence I'(p ∧ q) = 1 - and there is a bivalent extension I" of i for which I"(q) = 1 and I"(p) = 0 - hence I"(p ∧ q) = 0.

Example 3: i(p) = u; i(q) = 0
I(p ∧ q) = 0 because for all bivalent extensions I' of i, I'(q) = 0, hence I'(p ∧ q) = 0.

Example 4: i(p) = u; i(q) = u
I(p ∧ q) = # because there is a bivalent extension I' of i for which I'(p) = I'(q) = 1 - hence I'(p ∧ q) = 1 - and there is a bivalent extension I" of i for which I"(q) = I"(p) = 0 - hence I"(p ∧ q) = 0.

Example 5: i(p) = u; i(q) = u
I(p ∧ ¬q) = # (as in Example 4).

Example 6: i(p) = u
I(p ∧ ¬p) = 0 because for every bivalent extension I' of i, I'(I(p ∧ ¬p) = 0.

(4) **Main Properties**

a. The Strong Kleene Logic is compositional (by definition!)

a'. The simple supervaluationist logic is not compositional.
In Example 4 and in Example 5:
I(first conjunct) = I(second conjunct) = #.
However, I(p ∧ ¬q) = # while I(p ∧ ¬p) = 0.

b. Many statements which are tautologies or contradictions in classical logic can have the value # in the Strong Kleene Logic.
For instance, if I(p) = #, I(p ∧ ¬q) = # = I(p ∨ ¬q).

b'. If a statement is a tautology or a contradiction in classical logic, it has the same status in the supervaluationist logic.

(5) **Partial Equivalence**

If *F* contains at most one occurrence of any propositional or predicate letter, then the Strong Kleene and Supervaluationist logics agree on *F*.

Proof [by induction on the complexity of formulas].
We write i for a trivalent valuation function, K for its Strong Kleene extension, S for its supervaluationist extension.

If F contains at most one occurrence of any propositional or predicate letter, then $K_s(F) = S_s(F)$.

$F = p, P(x)$: trivial
$F = \neg G$: trivial
$F = (G \wedge G')$ By the induction hypothesis, $K_s(G) = S_s(G)$ and $K_s(G') = S_s(G')$.
-Case 1: $K_s(F) = 1$. Then $K_s(G) = K_s(G') = S_s(G) = S_s(G') = 1$, hence for every bivalent $I \geq i$, $I_s(G) = I_s(G') = 1$, hence for every bivalent $I \geq i$, $I_s(F) = 1$ and thus $S_s(F) = 1$.
-Case 2: $K_s(F) = 0$. Suppose $K_s(G) = 0$ (the other case is symmetric). Then $S_s(G) = 0$ and thus for every bivalent $I \geq i$, $I_s(G) = 0$. Therefore for every bivalent $I \geq i$, $I_s(F) = 0$, and thus $S_s(F) = 0$.
-Case 3: $K_s(F) = \#$.
(a) If $K_s(G) = 1$ and $K_s(G') = \#$ (or conversely - the reasoning is the same), then for every bivalent $I \geq i$, $I_s(G) = 1$ for some bivalent I', $I'' \geq i$, $I'_s(G) = 1$ and $I'_s(G') = 1$ and $I''_s(G) = 1$ and $I''_s(G') = 0$. Therefore for $S_s(F) = \#$.
(b) If $K_s(G) = K_s(G') = \#$, some bivalent $I, J, I', J' \geq i$, $I_s(G) = I'_s(G') = 1$, $J_s(G) = J'_s(G') = 0$. Let us call At the set of all atomic propositions and predicates, and At(F) the set of atomic propositions and predicates that occur in F. We can define:
$I+I' = I_{/(At-At(G'))} \cup I'_{/At(G')}$
$I+J' = I_{/(At-At(G'))} \cup J'_{/At(G')}$
$J+I' = J_{/(At-At(G'))} \cup I'_{/At(G')}$
$J+J' = J_{/(At-At(G'))} \cup J'_{/At(G')}$

Since $At(G) \cap At(G') = \emptyset$, $I+I'_s(G) = I_s(G)$ and $I+I'_s(G') = I'_s(G')$; and similarly for the other cases. It follows in particular that
$I+I'_s(G) = I+I'_s(G') = 1$, hence $I+I''_s(F) = 1$
$J+J'_s(G) = J+J'_s(G') = 0$, hence $J+J'_s(F) = 0$
and thus $S_s(F) = \#$.

$F = \forall x\ G$
Case 1. $K_s(F) = 1$. Then for every object d, $K_{s[x\to d]}(G) = 1$, and thus for every d, $S_{s[x\to d]}(G) = 1$. In other words, for every d, for every bivalent $I' \geq i$, $I'_{s[x\to d]}(G) = 1$. Hence for every bivalent $I' \geq i$, $I'_s(F) = 1$.Therefore $S_s(F) = 1$.
Case 2. $K_s(F) = 0$. Then for some object d, $K_{s[x\to d]}(G) = 1$, and thus for some d, $S_{s[x\to d]}(G) = 1$. Therefore for every bivalent $I' \geq i$, $I'_{s[x\to d]}(G) = 0$. So for every bivalent $I' \geq i$, $I'_s(F) = 0$.Therefore $S_s(F) = 1$.
Case 3. $K_s(F) = \#$. So (i) for some object d, $K_{s[x\to d]}(G) = \#$, (ii) for no object d, $K_{s[x\to d]}(G) = 0$. Therefore (i) for some object d, $S_{s[x\to d]}(G) = \#$, (ii) for no object d, $S_{s[x\to d]}(G) = 0$.
Thus (i) for some object d, for some bivalent I', $I'' \geq i$, $I'_{s[x\to d]}(G) \neq I''_{s[x\to d]}$, (ii) for every object d, for some bivalent $I' \geq i$, $I'_{s[x\to d]}(G) \neq 0$. By (i), $S_s(F) \neq 1$; by (ii), $S_s(F) \neq 0$. So $S_f(F) = \#$.