On the Logic of LGB Type Structures. Part II: Reentrancy Structures

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ABSTRACT. It is shown that the dynamic logics corresponding to theories of generative grammar are decidable. The theorems establish fairly general decidability results for structures that use a restricted form of reentrancy. Our methods give effective algorithms and upper bounds for the complexity of the decision problem. Although the bounds are fairly high (sometimes in the order 2EXPTIME) it is hoped that the complexity can be reduced in most cases. This opens the way to create effective tools for testing grammars and theories, and for checking satisfiability of web queries for reentrant structures.

1. INTRODUCTION

This paper is a continuation of [11]. Although the proofs are independent of those of the previous paper, we omit here the motivations that lead to the definitions of the structures whose logics we shall look at here.

A grammar can be seen as a theory of a class $\mathcal{K}$ of structures. A framework admits classes of grammars, and so classes of classes of structures. As I have shown in [9], the structures used in generative grammar since the GB framework—thus including the theory of Principles and Parameters as well as the Minimalist Program—are more complex than trees. Their geometrical part consists of a relational structure of the form $\langle M, \prec \rangle$, where M is finite, $\prec^+$ is cycle free, and for all x, $\{y : x \prec y\}$ is linearly ordered by $\prec^+$. These are called **multidominance structures**. Everything else, starting from categories, adjunction and subcategorisation can be dealt with by adding suitable propositional constants or by considering lexical entries to be structurally complex (see [11]). Adjunction complicates the matter insofar as it requires a more liberal notion of reentrancy. We shall deal at the end of the paper with adjunction by proving a very general theorem that admits structures that have any kind of relations added to trees. Thus, particular grammars determine classes of MDSs. Theories of grammar determine sets of grammars and thus—indirectly—sets of logics. Consider now a class of structures $\mathcal{K}$. We may now ask: what is the logic of these structures? In the literature one typically settles either on monadic second order logic (see [12]) or on modal logic. The disadvantage of MSO is that multidominance prohibits a straight application of Rabin's Theorem. So far it is—to my knowledge—not known, how Rabin's Theorem can be applied to MDSs. But it is not necessary to use such strong logics, whose complexity anyhow is far beyond the practical needs. It turns out that everything that needs to be said at all can be said in a relatively weak logic, using five modal operators. The language allows to distinguish two different grammars and is therefore expressive enough.

We shall show that virtually all theories formulated within GB and MP are decidable. This is a big step forward. It technically means that one can decide in principle whether a principle follows from a conjunction of other principles, whether a grammar admits certain types of structures, whether a given string is grammatical given a particular grammar, and so on. It means that these questions can be settled once and for all in a formal way, and not simply by looking at examples. For all those who do not wish to perform such arguments themselves it may be said that since all methods are constructive they can be executed by a machine as well and so it is also possible to design work benches for syntacticians that allows to define a grammar or even a constraint on grammars and see how it interacts with other constraints, or whether it is satisfied in a grammar, and so on. It also allows to check queries on linguistic structures for consistency with a given theory, eliminating costly searches in the internet in case of unsatisfiability.

2. Reentrancy Structures and Multidominance Structures

In the previous papers we studied the logic of MDSs. It turns out that MDSs can be coded in reentrancy structures. From a formal viewpoint, reentrancy structures are somewhat easier to deal with. A reentrancy structure uses two kinds of relations: the 'white' relations $\lhd_i$, $i < m$, and the 'black' relations $\blacktriangleright_j : j < n\}$. On the basis of these relations we set

(1) $$< := \left(\bigcup_{i<m} \lhd_i\right)^+$$

Definition 1. *A **reentrancy structure** is a structure*

(2) $$\mathfrak{M} = \langle M, \{\lhd_i : i < m\}, \{\blacktriangleright_j : j < n\}\rangle$$

*where $\langle M, \{\lhd_i : i < m\}\rangle$ is an m-branching tree with successor relations $\lhd_i$, $i < m$, and for every $j < n$, $\blacktriangleright_j$ is a partial function such that $\blacktriangleright_j \subseteq >$. $\mathfrak{M}$ is **narrow** if for all $j < n$, $\blacktriangleleft_j := \blacktriangleright_j^{\smile}$ also is a partial function.*

(3) $$\lhd := \bigcup_{i<m} \lhd_i$$

We use $\triangledown_i$ and $\triangledown$ for the operators that use these relations. And we use $\blacktriangledown_j$ for the operator of the relation $\blacktriangleright_j$.

Before we dive into technicalities let us indicate how we code MDSs by means of narrow reentrancy structures. Recall from [11] the following.

Definition 2. *A **multidominance structure** is a structure*

$$\langle M, \succ_{00}, \succ_{01}, \succ_{10}, \succ_{11} \rangle \tag{4}$$

such that

① $\langle M, \succ_{00}, \succ_{01} \rangle$ *is a tree with dominance relation* $(\succ_{00} \cup \succ_{10})^+$.
② *If there is a* $y \prec_{1i} x$ *for some* $i \in \{0, 1\}$ *then there is a* $y' \prec_{0j} x$ *for some* $j \in \{0, 1\}$.
③ *For* $i \in \{0, 1\}$*: If* $x \succ_{i1} y$ *then* $x\ (\succ_{00} \cup \succ_{10})^+\ y$.
④ *For* $i \in \{0, 1\}$*: If* $x \succ_{i0} y$ *and* $x \succ_{i1} y'$ *then* $y = y'$.
⑤ *For* $i \in \{0, 1\}$*:* $\succ_{i0} \cap \succ_{i1} = \varnothing$.
⑥ *For* $i \in \{0, 1\}$*:* $\succ_{0i} \cap \succ_{1i} = \varnothing$.

Let $M(x) := \{y : y \succ_{01} x \text{ or } y \succ_{11} x\}$. $\mathfrak{M}$ *is called **narrow** if* $|M(x)| \leq 1$ *for every* $x \in M$.

It follows from the definitions that

$$(\succ_{00} \cup \succ_{01} \cup \succ_{10} \cup \succ_{11})^+ = (\succ_{00} \cup \succ_{10})^+ \tag{5}$$

Let $\mathfrak{M}$ be given. For all $i \in \{0, 1\}$ we put

$$\lhd_i := \succ_i \tag{6}$$

Let $M(x) := \{y : x \succ_{\bullet 1} y\}$. By definition. $M(x)$ is linearly ordered by the tree order $<$. So, $M(x) = \{y_i : i < p\}$, where $y_k < y_{k+1}$ for all $k < p - 1$. Then put

$$\begin{aligned} y_i \blacktriangleleft_{0j} z &:\Leftrightarrow z = y_{i+1}, x \prec_{1j} y_i \text{ and } x \prec_{01} y_{i+1} \\ y_i \blacktriangleleft_{1j} z &:\Leftrightarrow z = y_{i+1}, x \prec_{1j} y_{i+1} \text{ and } x \prec_{11} y_{i+1} \end{aligned} \tag{7}$$

First of all, let us note the following.

Lemma 3. $\blacktriangleright_{ij}$ *and* $\blacktriangleleft_{ij}$ *are functions.*

Proof. Assume $x \blacktriangleleft_{00} y$ and $x \blacktriangleleft_{00} y'$. Then $x, y \in M(z)$ for some z, and $x, y' \in M(z')$ for some z'. Now, $z = z'$, otherwise $M(z) \cap M(z') = \varnothing$. To see this, observe that $x \in M(z) \cap M(z')$, which is to say that $x \succ z$ as well as $x \succ z'$. Moreover, by definition of $\blacktriangleleft_{00}$, $x \succ_0 z$ and $x \succ_0 z'$. This means $z = z'$. So, $y = y'$, since y and y' both are the next node up from x in the tree order. Similarly for the other relations $\blacktriangleleft_{ij}$. Now assume $x \blacktriangleright_{00} y$ and $x \blacktriangleright_{00} y'$. This again means $x, y \in M(z)$ for some z, $x, y' \in M(z')$ for some z', from which $z = z'$. Since $M(z)$ is linearly ordered by the tree order, $y = y'$, by definition of $\blacktriangleright_{00}$. Similarly for the other relations $\blacktriangleright_{ij}$. □

Put

(8) $$\nu(\mathfrak{M}) := \langle M, \{\rhd_i : i < 2\}, \{\blacktriangleright_{ij} : i, j < 2\}\rangle$$

It is not hard to see that $\blacktriangleright_{ij} \subseteq \rhd^+$ for all $i, j < 2$. Hence we have

Lemma 4. *$\nu(\mathfrak{M})$ is a narrow reentrancy structure.*

We have to see how to recover $\mathfrak{M}$ from $\nu(\mathfrak{M})$. Put

(9) $$\succ_{00} := \lhd_0$$

(10) $$\succ_{10} := \lhd_1$$

This defines the same tree order underlying the MDS. Put

(11) $$\blacktriangleleft_i := \blacktriangleleft_{i0} \cup \blacktriangleleft_{i1}$$

We define now: $x \succ_{10} y$ iff there is a $j < 2$ and a sequence $x = z_0 \lhd_j z_1 \blacktriangleleft_j z_2 \blacktriangleleft_j z_3 \cdots \blacktriangleleft_{j0} z_n = y$, $n > 1$. $x \succ_{11} y$ iff there is a $j < 2$ and a sequence $x = z_0 \lhd_j z_1 \blacktriangleleft_j z_2 \blacktriangleleft_j z_3 \cdots \blacktriangleleft_{j1} z_n = y$, $n > 1$. Given a reentrancy structure $\mathfrak{N}$ put

(12) $$\mu(\mathfrak{N}) := \langle M, \succ_{01}, \succ_{10}, \succ_{10}, \succ_{11}\rangle$$

This defines an MDS, a fact that follows immediately from the following fact.

Lemma 5. *For every MDS: $\mu(\nu(\mathfrak{M})) = \mathfrak{M}$.*

Proof. The tree order is identical, so we need to care only about $\succ_{10}$ and $\succ_{11}$. Suppose that in $\mathfrak{M}$ $x \succ_{10} y$. Then $y \in M(x)$ and so we can enumerate $M(x)$ as z_i, $i < p + 1$ such that $x \succ_{\bullet 0} z_1 \succ^+_{\bullet 0} z_2 \succ^+_{\bullet 0} z_3 \cdots \succ^+_{\bullet 0} z_p$. Suppose that $x \succ_{00} z_0$. Then $x \lhd_0 z_0$. By definition of $\blacktriangleleft_{ij}$, we have $z_i \blacktriangleleft_{0i} z_{i+1}$ for all $i < p$. Moreover, we have $z_{p-1} \blacktriangleleft_{00} z_p$. This is exactly the definition of (the reconstructed version of) $\succ_{10}$ given above. Similarly for the other cases. □

For the next theorem we only need to remark that we can use the full power of dynamic logic.

Corollary 6. *The modal logic of reentrancy structures has the finite model property and is decidable.*

3. Axiomatisation

We work with polymodal logic. We shall now rehearse the details, and refer instead to standard sources. Let Var $:= \{p_i : i \in \mathbb{N}\}$ be the set of variables. The set of constants is C. The boolean connectives are $\top$, $\neg$ and $\wedge$. For every white relation $\rhd_i$ we assume a modal operator $[\triangledown_i]$, and for every black relation $\blacktriangleright_j$ a modal operator $[\blacktriangledown_j]$. (In fact, $\triangledown_i$ and $\blacktriangledown_j$ are the programs in the sense of dynamic logic, and modalities are formed from them by using the brackets $[-]$ or $\langle - \rangle$.) Finally, there is a master modality

$\Box$. We also write $\boxdot\varphi := \varphi \wedge \Box\varphi$. The first set of postulates regulates that the structure $\langle M, \{\rhd_i : i < m\}\rangle$ is acyclic.

(R1) $\Box(\Box p \to p) \to \Box p$.
(R2) For all $i < m$: $\langle\triangledown_i\rangle p \to \Diamond p$.
(R3) For all $i < m$: $\langle\triangledown_i\rangle p \to [\triangledown_i]p$.

The proof is not repeated here. Using unravelling one can show that the structures all derive from finite trees by collapsing certain subtrees.

The logic with the axioms (R1) – (R3) is the logic of acyclic structures. We call it $\mathbf{AC_m}$. If we expand the language into dynamic logic we get the logic $\mathbf{DPDL_m.f}$, also called the logic of finite deterministic computations on m programs. This logic was shown in [8] to be complete with respect to finite m-branching trees. Although we are not dealing with a dynamic logic, it was proved in [11], that the star can be added with impunity, since all programs are terminating in finite structures. Technically, therefore, although the language defined above contains no programs in the sense of PDL, we can add them in the form of abbreviations in the following way.

(13)
$$\begin{aligned}\langle\alpha\cup\beta\rangle\chi &:= \langle\alpha\rangle\chi \vee \langle\beta\rangle\chi\\ \langle\alpha;\beta\rangle\chi &:= \langle\alpha\rangle\langle\beta\rangle\chi\\ \langle\delta?\rangle\chi &:= \delta \wedge \chi\\ \langle\alpha^*\rangle\chi &:= \boxdot(q \leftrightarrow \chi \vee \langle\alpha\rangle\chi) \to q\end{aligned}$$

where in the last line q is a variable not occurring in χ or α and α is cycle free. The last reduction works in general for every program, since $R(\alpha)$ can be shown in **PDL.f** to be always of the form $R(\delta?) \cup R(\beta)$ for a cycle free β. To be a bit more precise, let us be given a formula φ in the language of **DPDL.f**. Then for every formula χ in the Fisher-Ladner closure we introduce a variable q_χ and replace φ by

(14)
$$\boxdot D(\varphi) \to q_\varphi$$

where $D(\varphi)$ is the conjunction of the following formulae:

(15)
$$\begin{aligned}q_{\langle\alpha;\beta\rangle\delta} &\leftrightarrow q_{\langle\alpha\rangle\langle\beta\rangle\delta}\\ q_{\langle\alpha\cup\beta\rangle\delta} &\leftrightarrow q_{\langle\alpha\rangle\delta} \vee q_{\langle\beta\rangle\delta}\\ q_{\langle\eta?\rangle\delta} &\leftrightarrow q_\eta \wedge q_\delta\\ q_{\langle\alpha^*\rangle\delta} &\leftrightarrow q_\delta \vee q_{\langle\alpha;\alpha^*\rangle\delta}\\ q_{\langle\triangledown_i\rangle\delta} &\leftrightarrow \langle\triangledown_i\rangle q_\delta\\ q_{\Diamond\delta} &\leftrightarrow \Diamond q_\delta\end{aligned}$$

Moreover, as we have explained in [11], it is possible to add the converse $\alpha^\smile$ with impunity provided that $R(\alpha^\smile)$ is a partial function and that we have a formula c equivalent to $[\alpha^{\smile}]\bot$. In order to replace $\alpha^{\smile}$, suppose the following holds at the root of the frame:

$$\boxdot(\chi \leftrightarrow (\chi \wedge c) \vee \langle\alpha\rangle q) \tag{16}$$

Then q is true at a node u iff $[\alpha^{\smile}]\chi$. Then $\langle\alpha^{\smile}\rangle\chi \leftrightarrow q \wedge \neg c$, by functionality of $\alpha^{\smile}$ and the definition of c. Alternatively, add to $D(\varphi)$ the following:

$$q_\delta \leftrightarrow q_\delta \wedge q_c \vee q_{\langle\alpha;\alpha^{\smile}\rangle\delta} \tag{17}$$

(This will require adding $q_{\langle\alpha;\alpha^{\smile}\rangle\delta}$ to the set of variables, and some more variables for the subformulae, whenever $\langle\alpha^{\smile}\rangle\delta$ is in the Fisher-Ladner closure.)

What we need, however, is the constant c. If, for example, we are interested in inverting the dominance relation, $\rhd$, we can do the following. If φ is consistent, so is $\varphi \wedge \Box\neg\varphi$. Thus, we can always assume that our model satisfies φ only at the root. In that case, the desired c is φ itself. It is likewise possible to invert $\succ_i$ for all $i < m$.

Thus, adding the converse does not increase expressibility as long as one can define c and the converse is a partial function. It follows that for every regular expression formed from programs which have this property, is also definable. However, one may not to use $*$ in programs that contain basic programs as well as their converses, since the resulting program may contain cycles.

The next batch of postulates concerns the axiomatisation of reentrancy structures.

(R4) For all $j < n$: $\langle\blacktriangledown_j\rangle p \rightarrow \Diamond p$.
(R5) For all $j < n$: $\langle\blacktriangledown_j\rangle p \rightarrow [\blacktriangledown_j]p$.
(R6) For all $j < n$: $\langle\blacktriangledown_j\rangle p \rightarrow \langle\triangledown^+\rangle p$.
(R7) For all $i < j < m$: $\langle\triangledown_i^{\smile}\rangle\top \rightarrow \neg\langle\triangledown_j^{\smile}\rangle\top$.

The axioms (R4) – (R5) are unproblematic, see the discussion above. They in fact give us the logic of finite computations for $m+n$ programs. (R6) adds the requirement that $\rhd_i$ is contained in the closure of the $\lhd_i$. (R7) finally makes the models trees. For it says that any node can have only one mother via $\lhd$ so that reentrancy is eliminated for the white relations. Notice that we have used the converse here; so although this postulate seems to use no variables, it abbreviates a formula that does.

Finally, we present an axiom for narrow reentrancy structures. Before we begin we draw attention to the fact that in trees one can effectively use nominals; these are variables which are true at exactly one point (see [1]). Set

$$n(p) := \neg \boxdot \neg p \wedge \boxdot(p \rightarrow [\triangledown^+]\neg p) \wedge \boxdot\neg \bigwedge_{i<j<m} (\langle\triangledown_i; \triangledown^*\rangle p \wedge \langle\triangledown_j; \triangledown^*\rangle p) \tag{18}$$

Lemma 7. *Suppose* $\mathfrak{M} = \langle M, \{\rhd_i : i < m\}\rangle$ *is a tree with root* w. *Then* $\langle \mathfrak{M}, \beta, w\rangle \vDash n(p)$ *iff* $\beta(p) = \{x\}$ *for some* $x \in M$.

Proof. (⇒). First, $w \vDash \neg \boxdot \neg p$ guarantees that the set $\beta(p)$ is nonempty. Next, since $w \vDash \boxdot(p \to [\triangledown^+]\neg p)$ and w is the root, for every $x \in M$: $x \vDash p \to [\triangledown^+]\neg p$. This means that the set $\beta(p)$ is an antichain with respect to $\rhd^+$. For if $x \rhd^+ y$ and $x \vDash p$ then $y \vDash p$ cannot hold. Finally, this antichain has only one point. For if it contains two different points y and y' then there is a z and u, u' such that $y \lhd^* u \lhd_i z$ and $y' \lhd^* u' \lhd_j z$ for $i < j$. Thus $z \vDash \langle \triangledown_i; \triangledown^*\rangle p; \langle \triangledown_j; \triangledown^*\rangle p$, which is also excluded. (⇐). Basically similar. □

(R8) For all $j < n$: $n(p) \to \boxdot(\langle \blacktriangledown_j\rangle p \to [\triangledown^+; \blacktriangledown_j]\neg p)$

The logic **RS** the logic axiomatised over K_{m+n+1} by (R1) – (R7), while **NRS** is the logic obtained by adding to K_{m+m+1} the axioms (R1) – (R8).

Lemma 8. *A reentrancy structure is narrow iff it satisfies the postulate (R8).*

Proof. (⇐). Suppose $\mathfrak{M}$ is not narrow; say, $x \blacktriangleleft_{00} y, y'$ with $y \rhd^+ y'$. Define $\beta(p) := \{x\}$ and the above axiom is violated at y (and, as is not hard to see, also at the root). From Lemma 7 follows that at the root w:

(19) $$\langle \mathfrak{M}, \beta, w\rangle \vDash n(p)$$

Nevertheless,

(20) $$w \nvDash \boxdot(\langle \blacktriangledown_j\rangle p \to [\triangledown^+; \blacktriangledown_j]\neg p)$$

For we have $y \vDash \langle \blacktriangledown_j\rangle p$. However, $y \nvDash [\triangledown^+; \blacktriangledown_j]\neg p$ since $y \rhd^+ y'$ and $x \blacktriangleleft_j y'$ whence $y' \vDash \langle \blacktriangledown_j\rangle p$. (⇒). Now, conversely, assume that $\mathfrak{M}$ is narrow. Assume that β is such that

(21) $$\langle \mathfrak{M}, \beta, w\rangle \vDash n(p)$$

Then by Lemma 7, $\beta(p) = \{x\}$ for some x. Let y be such that

(22) $$y \vDash \langle \blacktriangledown_j\rangle p$$

Then $y \rhd^* x$, since p is only true at x. Let $y' \lhd^+ y$ be such that $y' \vDash \langle \blacktriangledown_j\rangle p$. Then $y' \rhd x$. However, also $y \rhd x \vDash p$. Since the structure is a narrow RS, $y = y'$. Contradiction. Hence, $y' \vDash \neg\langle \blacktriangledown_j\rangle p$, and since y' was arbitrary,

(23) $$y \vDash [\triangledown^+; \blacktriangledown_j]\neg p$$

as promised. □

We are ultimately interested in the logic of the class of structures of the form $\nu(\mathfrak{M})$. It is obtained by adding to **NRS** a few postulates. Namely, notice first that the relations $\rhd_0$, $\blacktriangleright_{00}$ and $\blacktriangleright_{10}$ are mutually exclusive. A node can only have one left daughter. Similarly for right hand daughters. And finally, if a node has a right hand daughter it also has a left hand daughter:

(R9) For all $i < 2$: $(\langle\triangledown_i\rangle\top \rightarrow \neg\langle\blacktriangledown_{0i}\top \wedge \neg\langle\blacktriangledown_{1i}\rangle\top) \wedge \langle\blacktriangledown_{0i}\rangle\top \rightarrow \neg\langle\triangledown_i\top \wedge \neg\langle\blacktriangledown_{1i}\rangle\top) \wedge \langle\blacktriangledown_{1i}\rangle\top \rightarrow \neg\langle\triangledown_i\top \wedge \neg\langle\blacktriangledown_{0i}\rangle\top)$

(R10) $(\langle\blacktriangledown_{01}\rangle\top \vee \langle\triangledown_1\top \vee \langle\blacktriangledown_{11}\rangle\top) \rightarrow (\langle\blacktriangledown_{00}\rangle\top \vee \langle\triangledown_0\top \vee \langle\blacktriangledown_{10}\rangle\top)$

It is clear that these axioms characterise the described properties; also, they are constant and therefore will be left out of consideration in the sequel, since constant axioms preserve decidability and also complexity.

In what is to follow we shall give a proof that both **RS** and **NRS** have the finite model property, and so characterise exactly the class of finite (narrow) RSs.

4. Some Basic Results on Complexity

Call a finite structure **linearisable** if there exists a linear irreflexive order $\ll$ computable in linear time such that (1) for every modal operator $\blacksquare$, $R(\blacksquare) \subseteq\ll$, and for every modal operator $\blacksquare$ either (2a) there is a number k such that $\blacksquare$ no point has more than k successors via $\blacksquare$, or (2b) there is a number k such that for every point x, $|\{y : x \ll y, \neg(x\ R(\blacksquare)\ y)\}| \leq k$.

Lemma 9. *Truth in a linearisable model is computable in linear time.*

Proof. Suppose that we have a well-order $\ll$ on the domain of the model such that if $x\ R(\blacksquare)\ y$ then $x \ll y$ for all basic modalities $\blacksquare$. Let $h(w) := |\{z : w \ll z\}|$. Assume that for all subformulae δ of φ and all $z \gg w$, truth at z is computed. Truth at w of a subformula δ is a matter of checking a bounded boolean combination of formulae. □

This can be applied to our logics as follows. Each node is given an address in the following way. The root has address ε. If $x \lhd_j y$ and y has address $\vec{c}$, then x has address $\vec{c} \cdot j$. By assumption, every node has a unique address. As order we take $\vec{c} \ll \vec{e}$ iff $\vec{c}$ is a prefix of $\vec{c}$ or there are $\vec{p}$, $\vec{q}$ and $\vec{r}$ such that $\vec{c} = \vec{p} \cdot 0 \cdot \vec{q}$ and $\vec{e} = \vec{p} \cdot 1 \cdot \vec{r}$. This takes care of the basic modalities. $\square$ still is tricky. Notice, however, that for a node $\vec{c}$, $\vec{c} \vDash \square\varphi$ iff $\vec{c} \cdot 0 \vDash \varphi; \square\varphi$, $\vec{c} \cdot 1 \vDash \varphi; \square\varphi$, so that truth at $\vec{c}$ is a bounded boolean combination of truth at some nodes later in the order.

The following is from [13].

Theorem 10 (Volper & Vardi)**.** *Satisfiability in* **DPDL.f** *is globally (and locally) EXPTIME-complete.*

5. Preliminaries

Let $\mathfrak{M}$, β, φ be fixed. We assume that φ contains only basic modalities; everything else is just an abbreviation as defined above. Let $\mathrm{SF}(\varphi)$ denote the set of subformulae of φ. For $A \subseteq \mathrm{SF}(\varphi)$ put

(24) $$a_{\mathfrak{M},\beta}(H) := \bigwedge_{\chi \in A} \chi \wedge \bigwedge_{\chi \in \mathrm{SF}(\varphi) - A} \neg\chi$$

If $\mathfrak{M}$ and β are clear from the context, we drop them and write $a(H)$. Such formulae are called φ-**atoms**. Let $\mathrm{At}(\varphi)$ denote the set of all φ-atoms. For a set Δ of formulae, the notation $\mathrm{SF}(\Delta)$ and $\mathrm{At}(\Delta)$ are used in the obvious way. For $w \in M$, let $a(w)$ denote the atom which is true at w.

Here is a general result on how to make new models from old ones. Fix a set Δ closed under taking subformulae. Let $\langle \mathfrak{M}, \beta \rangle$ be a model and $x, y \in M$. Write $x \sim_\Delta y$ if for all $\delta \in \Delta$: $\langle \mathfrak{M}, \beta, x \rangle \vDash delta$ iff $\langle \mathfrak{M}, \beta, y \rangle \vDash \delta$.

Lemma 11. *Let $\langle M, \{\lhd_i : i < m\}\rangle$ be a frame, β a valuation. Now let $N \subseteq M$ and $\hat{\lhd}_i$ be relations such that the following holds for all $i < m$, $x \in M$ and $z \in N$:*

① *If $x \lhd_i y$ then there exists a $y' \sim_\Delta y$ such that $y' \in N$ and $x \hat{\lhd}_i y'$.*
② *If $z \hat{\lhd}_i y$ then there exists a $y' \sim_\Delta y$ such that $x \lhd_i y'$.*

Then for all $y \in N$ and $\delta \in \Delta$:

(25) $$\langle\langle M, \{\lhd_i : i < n\}\rangle, \beta, y\rangle \vDash \delta \quad \Leftrightarrow \quad \langle\langle N, \{\hat{\lhd}_i : i < n\}\rangle, \beta, y\rangle \vDash \delta$$

Proof. By induction on the complexity of δ. If δ is a variable, the claim is trivial. The induction steps for $\neg$ and $\wedge$ are immediate. Now let $\delta = \Diamond_i \vartheta$. Then $\vartheta \in \Delta$, by assumption. ($\Rightarrow$). Assume that $\langle \mathfrak{M}, \beta, y\rangle \vDash \Diamond_i \vartheta$. Then there exists a $z \in M$ such that $y \lhd_i z$ and $\langle \mathfrak{M}, \beta, z\rangle \vDash \vartheta$. By assumption there is a $z' \sim_\Delta z$ such that $z' \in N$ and $y \hat{\lhd}_i z'$. Hence $\langle \mathfrak{M}, \beta, z'\rangle \vDash \vartheta$. By inductive hypothesis, $\langle \mathfrak{N}, \beta, z'\rangle \vDash \vartheta$ and so $\langle \mathfrak{N}, \beta, y\rangle \vDash \Diamond_i \vartheta$. ($\Leftarrow$). Assume now that $\langle \mathfrak{N}, \beta, y \vDash \Diamond_i \vartheta$. Then there is a $z \in N$ such that $y \hat{\lhd}_i z$ and $\langle \mathfrak{N}, \beta, z\rangle \vDash \vartheta$. By assumption, there is a $z' \sim_\Delta z$ such that $y \lhd_i z'$. By inductive hypothesis, $\langle \mathfrak{M}, \beta, z\rangle \vDash \vartheta$ and so $\langle \mathfrak{M}, \beta, z'\rangle \vDash \vartheta$. From this we get $\langle \mathfrak{M}, \beta, y\rangle \vDash \Diamond_i \vartheta$. □

Call w **minimal** if $w \vDash a(w) \wedge [\triangledown^+]\neg a(w)$. No two minimal points with same atom are comparable via $<$. Equivalently, if $a(w) = a(v)$ and v and w are both minimal, and $v \leq w$ then $w = v$. We shall show how to build a model on some set of minimal points. First, observe that we may choose the root of the model to be minimal. Let the **depth** of w be the defined by

(26) $$\mathrm{dp}(w) := |\{a(x) : x < w\}|$$

Call w **egregious** iff either (a) $\mathrm{dp}(w) = 0$ and w is the root and minimal, or (b) $\mathrm{dp}(w) = n + 1 > 0$ and for the unique x such that $w < x$ and $\mathrm{dp}(x) = n$, and y such that $w \leq y \lhd x$ w is the leftmost minimal member of the set

(27) $$\{z : z \leq x, a(z) = a(y)\}$$

(Here leftmost is defined as follows: if $x \lhd_i y$ and $x' \lhd_j y$ then x is left of x' iff $i < j$. In general, x is left of x' iff there are u, u' and z such that $x \leq u \lhd_i z$ and $x' \leq u' \lhd_j u$ and $i < j$.)

The definition makes sure that for every egregious x which has a daughter, there is a unique egregious w with depth $\mathrm{dp}(x) + 1$ below that daughter. We denote by x^{ε} the unique egregious point u such that $u \leq x$ and $a(u) = a(x)$.

We shall prove a rather general theorem on the logic of finite trees based on m primitive relations and one master $\Box$.

Theorem 12. *For every* $\mathbf{RS}$*-model* $\langle \mathfrak{M}, \beta, x\rangle \vDash \varphi$ *where* $<$ *is a tree order, there is an* $\mathbf{RS}$*-model for* φ *such that* $<$ *is a tree order, and which has at most* 2^{2^n} *points.*

Proof. Assume that $\langle \mathfrak{M}, \beta, w\rangle \vDash \varphi$. Let $E := \{x^{\varepsilon} : x \in M\}$ be the set of egregious points of $\langle \mathfrak{M}, \beta\rangle$. For a relation R put $\hat{R} := \{\langle x, y^{\varepsilon}\rangle : \langle x, y\rangle \in R\}$. Put

$$\mathfrak{E} := \langle E, \{\hat{\triangleright}_i \cap E^2 : i < m\}, \{\hat{\blacktriangleright}_j \cap E^2 : j < n\}\rangle \tag{28}$$

It is not hard to see that the order $\hat{>}$ is a tree order. First we notice that for egregious points y and y': $y \hat{<} y'$ iff $y < y'$. This is shown by induction on the number of egregious points between y and y'. Suppose this number is zero. ($\Rightarrow$). We have $y \hat{\triangleleft} y'$. Hence $y = x^{\varepsilon}$ for some $x \triangleleft y'$. From this follows $y < y'$. ($\Leftarrow$). $y < y'$ and there is x such that $y = x^{\varepsilon} \triangleleft y'$ from which $y \hat{\triangleleft} y'$. Now suppose that this number is not zero. ($\Rightarrow$). $y \hat{\triangleleft}^{+} y'$ implies $y < y'$ from the previous and the fact that $<$ is transitive. ($\Leftarrow$). There is a chain of egregious points $y = y_0 < y_1 < y_2 < \cdots < y_n = y'$ such that there is no egregious point between y_i and y_{i+1}, $i < n - 1$. By the previous, $y_i \hat{<} y_{i+1}$, and so $y \hat{<} y'$.

For assume that $y, y' \hat{>} x$. Then also $y, y' > x$, by the fact that $x^{\varepsilon} \leq x$ so that in general $y \hat{\triangleright}_i z$ implies $y > z$. It follows that $y \leq y'$ or $y' \leq y$. From this we get $y \hat{\leq} y'$ or $y' \hat{\leq} y$.

The valuation is $\gamma(p) := \beta(p) \cap E$. From Lemma 11 we get for every $\chi \in \mathrm{SF}(\varphi)$ that

$$\langle \mathfrak{E}, \gamma, x\rangle \vDash \chi \quad \Leftrightarrow \quad \langle \mathfrak{M}, \beta, x\rangle \vDash \chi \tag{29}$$

It follows that

$$\langle \mathfrak{E}, \gamma, w^{\varepsilon}\rangle \vDash \varphi \tag{30}$$

Thus the model is based on a tree. Given a formula of length n, there are n subformulae, whence 2^n atoms. The depth of a point is therefore bounded by 2^n. This is equal to the depth in $\mathfrak{E}$. It follows that since the models are based on binary branching trees of height at most 2^n there are at most 2^{2^n} points in E. $\Box$

This is the basis of all the proofs that we shall give in the sequel. The only addition they make is that there shall be additional relations to take care of.

6. The Main Theorem

We now formulate our main theorem.

Theorem 13. **NRS** *has the finite model property, and the size of models is bounded from above by* 2^{2^n}, *where n is the length of the formula.*

Proof. Let φ be given. We write $a(v)$ for the φ-atom of v. For each $\alpha \in \mathrm{At}(\varphi)$ let r_α be a new variable and put $\mathrm{MVar} := \{r_\alpha : \alpha \in \mathrm{At}(\varphi)\}$. Next let

$$\begin{array}{rll} \Pi := & & \bigwedge\langle \neg r_\alpha : \alpha \in \mathrm{At}(\varphi)\rangle \\ & \wedge & \bigwedge\langle \boxdot[\triangledown^+]\neg r_\alpha : \alpha \in \mathrm{At}(\varphi)\rangle \\ & \wedge & \bigwedge\langle \boxdot[\blacktriangledown_j](\alpha \to r_\alpha) : \alpha \in \mathrm{At}(\varphi), j < n\rangle \\ & \wedge & \bigwedge\langle \boxdot(r_\alpha \to \neg r_\beta) : \alpha \neq \beta \in \mathrm{At}(\varphi)\rangle \end{array} \tag{31}$$

First we verify that if $\langle \mathfrak{M}, \beta, w\rangle \vDash \varphi$, where $\beta : \mathrm{Var} \to \wp(M)$ is a valuation and $<$ a tree order on $\mathfrak{M}$ then there is a unique valuation β^+ on $\mathrm{Var} \cup \mathrm{MVar}$ extending β such that $\langle \mathfrak{M}, \beta^+, w\rangle \vDash \varphi; \Pi$. The formulae say the following. (1) r_α is not true at the root. (2) r_α is not true at a node x where there is a y such that $x < y$. Hence, r_α can only be true if there is a y such that $x \blacktriangleleft_j y$. (3) says that in that case r_α is true iff α is true at x. Since $\mathfrak{M}$ is a tree, y is unique, and so the valuation is uniquely defined. (4) says that not both r_α and r_β can hold if $\alpha \neq \beta$. (It is actually a consequence of (1), (2) and (3).) So,

$$\beta(r_\alpha) = \{u : a(u) = \alpha \text{ and for no } v\text{: } u \lhd v\} \tag{32}$$

Suppose that φ is **DPDL.f**-consistent and φ contains no variable from MVar. Then $\varphi; \Pi$ is **DPDL.f**-consistent as well.

Define

$$\begin{array}{rl} X(\varphi) := & \bigwedge\langle n(r_\alpha) \to \boxdot(\bigwedge\langle\blacktriangledown_j\rangle r_\alpha \to [\triangledown^+; \blacktriangledown_j]\neg r_\alpha) \\ & \quad : \alpha \in \mathrm{At}(\varphi), j < n\rangle \\ & \wedge \bigwedge\langle\langle\blacktriangledown_j\rangle\pi \to \langle\triangledown^+\rangle\pi : \pi \in \mathrm{At}(\varphi), j < n\rangle \end{array} \tag{33}$$

Notice that $X(\varphi)$ is a set of instances of **NRS**-axioms.

Now suppose that φ is **NRS**-consistent. Hence $\varphi; X(\varphi)$ is **NRS**-consistent and a fortiori **DPDL.f**-consistent. Thus, $\varphi; \Pi; X(\varphi)$ is **DPDL.f**-consistent. Therefore there is a structure $\mathfrak{T} := \langle T, \{\rhd_i : i < m\}, \{\blacktriangleright_j : j < n\}\rangle$ which is an $m + n$-branching tree, a valuation β and a world w such that

$$\langle \mathfrak{T}, \beta, w\rangle \vDash \varphi; \Pi; X(\varphi) \tag{34}$$

and

❶ all $\lhd_i$, $i < m$, and all $\blacktriangleleft_j$, $j < n$, are partial functions;
❷ all $\rhd_i$, $i < m$, and all $\blacktriangleright_j$, $j < n$, are partial functions, and

❸ $\lhd$ is cycle free.

This is because **DPDL.f** has the finite model property and the fact that we can apply unravelling. We shall now produce a narrow reentrancy structure based on the egregious points.

Let H be the closure of w under the relation $>$. Define the function $u \mapsto u^{\varepsilon}$ as in the proof of the previous theorem. Then let $E := \{x^{\varepsilon} : x \in H\}$ be the set of egregious points and define $\hat{\rhd}_i$, $i < n$ as before. Now suppose that $x \blacktriangleright_j u$, $j < n$. Then $\langle \mathfrak{T}, \beta, x\rangle \vDash \langle \blacktriangledown_j\rangle a(u)$. So, by choice of $X(\varphi)$, we have $\langle \mathfrak{T}, \beta, x\rangle \vDash \langle \triangledown^+\rangle a(u)$. So we choose an egregious $u^{\heartsuit}$ such that $u^{\heartsuit} < x$ and $a(u^{\heartsuit}) = a(u)$. Then $u^{\heartsuit} \in E$. We keep the partial function $u \mapsto u^{\heartsuit}$ fixed now. (In fact, a single such function suffices.)

$$\mathfrak{E} := \langle E, \{\hat{\rhd}_i : i < m\}, \{\hat{\blacktriangleright}_j : j < n\}\rangle \tag{35}$$

The valuation is $\gamma(p) := \beta^+(p) \cap E$. For $\chi \in \mathrm{SF}(\varphi)$ we get by Lemma 11 that for every egregious x:

$$\langle \mathfrak{E}, \gamma, x\rangle \vDash \chi \quad \Leftrightarrow \quad \langle \mathfrak{T}, \beta^+, x\rangle \vDash \chi \tag{36}$$

Thus we have a model $\langle \mathfrak{E}, \gamma, w^{\varepsilon}\rangle \vDash \varphi$. Finally, we need to see that $\mathfrak{E}$ is a narrow reentrancy structure. Recall the function $u \mapsto u^{\heartsuit}$. By definition, $u^{\heartsuit} < x$ if $u \lhd x$, and from $u^{\heartsuit} < x$ it follows that $u^{\heartsuit} \hat{<} x$. Thus, $\mathfrak{E}$ is a reentrancy structure. To show that it is narrow we need to show that the map $u \mapsto u^{\heartsuit}$ is injective. Let therefore u and v be distinct egregious points such that $u^{\heartsuit} = v^{\heartsuit}$. Let $u \blacktriangleleft_j x$ and $v \blacktriangleleft_j y$. Then, since $u^{\heartsuit} < x$ and $v^{\heartsuit} < y$ we have $v^{\heartsuit} < x, y$ and so $y \leq x$ or $x \leq y$. Without loss of generality, assume the first. Then either $x = y$, and we are done; or $y < x$. We shall derive a contradiction. Notice that $a(u) = a(v)$, whence u and v both satisfy the same r_α in $\mathfrak{T}$. Since $\langle \mathfrak{T}, \beta, w\rangle \vDash n(r_\alpha)$ and $\langle \mathfrak{T}, \beta^+, x\rangle \vDash \langle \blacktriangledown_j\rangle r_\alpha$, we find that $\langle \mathfrak{T}, \beta^+, x\rangle \vDash [\triangledown^+; \blacktriangledown_j]\neg r_\alpha$, in particular $\langle \mathfrak{T}, \beta^+, y\rangle \vDash \langle \blacktriangledown_j\rangle \neg r_\alpha$. This is the desired contradiction. □

Theorem 14. *Satisfiability in* **NRS** *is decidable in 2EXPTIME.*

Proof. Let n be the length of the formula. $\mathrm{SF}(\varphi)$ is linear in n, and so there are 2^n many atoms. Observe next that the auxiliary formulae are of combined length $O(2^n)$. This is because there are 2^n atoms, and there are 2^{n^2} formulae of the form $r_\beta \to \neg r_\alpha$ and $c2^n$ formulae of the remaining kinds. Notice, though, that the formulae of the first kind are redundant. So, we really only need $c2^n$ many formulae. Each formula has length at most dn for some d. This, combined with the fact that the logic of trees is EXPTIME, gives the result. □

7. Further Results: Distance Principles

In [11] I have considered variants of the definition of MDSs where lengths of movement steps are restricted. We shall look at such principle here again. Since we have changed the format of encoding, the form of the distance principles also changes slightly. Notice that in the reentrancy format it is not the links that get encoded directly but rather the movement paths. So, if $x \blacktriangleright_{ij} y$ this means that x and y are members of a chain and that there has been movement from y to x. This has advantages in the codification of movement. For we can set down distance principles in a very direct way as follows. While before we had

$$\langle \blacktriangledown_j \rangle p \to \langle \triangledown^+ \rangle p \tag{37}$$

we now consider postulates of the form

$$\langle \blacktriangledown_j \rangle p \to \langle \alpha_j \rangle p \tag{38}$$

There are now two cases to consider. If we consider Freeze Movement then the distance covered in a single movement step is measured in terms of underived links (see [10]), that is, white relations. We can capture this by requiring that α_j is a program not using any of the $\blacktriangledown_j$. If we are interested in Shortest Move then matters are different. Here the movement path will involve also derived links, that is to say, black relations. In this case the principle is stated as follows:

$$\langle \blacktriangledown_j \rangle p \to \langle \alpha_j \rangle p \tag{39}$$

with the condition that in the execution chain (to be defined below) δ_0 does not contain $\blacktriangledown_j$. This condition ensures that we measure the movement path of the link against the different alternatives. We shall defer the treatment of Shortest Steps and concentrate here on Freeze derivations.

On certain conditions on α_j the present construction can be repeated almost verbatim. What one must ensure is that if $y \xrightarrow{\alpha_j} x$ holds in $\mathfrak{T}$, it also holds in $\mathfrak{E}$. This is not the case for all α_j. However, under certain conditions this is the case. One case that interests us here is the case where α_j defines a command relation in terms of the white relations (see [8]).

We shall give a proof below. Command relations have the property that they continue to hold even if points are removed in a tree. To define that notion, let us say the following. An **execution chain** of α is a series $\gamma = \pi_0; \delta_0?; \pi_1; \delta_1?; \cdots; \pi_{n-1}; \delta_{n-1}?$ such that all π_i, $i < n$, are basic programs and $R([\gamma]) \subseteq R([\alpha])$. γ' is called a **subchain** if it is obtained from γ by removing some occurrences of the δ_i or π_i (but keeping their order).

Definition 15. *Let α be a program. α has the* ***subchain property*** *if for every execution chain γ of α every subchain of γ is an execution chain of α as well.*

Let us compare the chains of programs in $\mathfrak{E}$ and $\mathfrak{M}$, in particular those of the tree relations. If $x \rhd_i y$ in $\mathfrak{E}$ then $x \succ_i y'$ for some $y' \geq y$ with $a(y') = a(y)$. Thus, all we can say is the following. Every chain $\gamma = \triangledown_j; \delta?$ in $\mathfrak{E}$ where $\delta? \in \mathrm{SF}(\varphi)$ corresponds to an execution chain γ' of the program

$$\triangledown_j; \delta? \cup \triangledown_j; \delta?; (\triangledown; \top?)^*; \triangledown; \delta? \tag{40}$$

It is not hard to see that γ is a subchain of γ'.

Definition 16. *Let α be a program that has the subchain property. A reentrancy structure is called (α, j)-**distance restricted** if the logic satisfies*

$$\langle \blacktriangledown_j \rangle p \to \langle \alpha \rangle p \tag{41}$$

*For $\Delta = \{(\alpha_0, 0), (\alpha_1, 1), \cdots, (\alpha_{n-1}, n-1)\}$ we say that $\mathfrak{M}$ is Δ-**distance restricted** if it is (α_i, i)-distance restricted for every $i < n$.*

Now recall again the proof of Theorem 13. The proof goes through as before. What we must ensure however is that $\mathfrak{E}$ also is a structure for the logic. To this end it suffices to note the following: every chain of α in $\mathfrak{E}$ is a subchain of a chain of α in $\mathfrak{M}$. By construction, if $x \blacktriangleleft_j y$ in $\mathfrak{M}$ there is a z such that $z\ R([\alpha])\ y$ and $x = z$ or $x \blacktriangleleft_j z$. We have seen to it that there is also an egregious z such that either $z = x$ or $x \blacktriangleleft_j z$. What remains to be seen is that $z\ R([\alpha])\ y$ in $\mathfrak{E}$. This follows from the fact that there is an α-chain γ in $\mathfrak{M}$ from y to z, and it has a correlate $\gamma\ \mathfrak{E}$, which is a subchain of γ. Hence it is an α-chain from y to z, as promised.

Theorem 17. *For every Δ where all programs have the subchain property the logic of Δ-distance restricted reentrancy structures has the finite model property and is decidable in 2EXPTIME.*

8. Movement

We shall point a particular application. A **command relation** is a relation R that is characterised by the following property: there is a finite set S of sequences $\vec{\eta} = \langle \eta_0; \eta_1; \cdots; \eta_{n-1} \rangle$ of constant formulae such that $x\ R\ y$ iff for the least z that strictly dominates x and nonstrictly dominates y: the sequence of points that are strictly between x and z does not contain a subsequence that is contained in S. Somewhat more exactly: S contains sequences of properties of points, and a sequence $\langle x_i : i < n \rangle$ of points satisfies such a sequence $\vec{\eta}$ iff $x_i \vDash \eta_i$ for all $i < n$. Let us denote the relation by $C(S)$.

For example, **idc-command** is defined be the set $\{\langle\top\rangle\}$. Thus, x c-commands y iff for the least $z > x$ and $z \geq y$: the set $U = \{u : x < u < z\}$ does not contain a subsequence satisfying $\langle\top\rangle$. This is the case iff there is no nonempty subsequence iff $U = \varnothing$ iff z is immediately above x. Next, **0-subjacency** is defined by $\{\langle \mathrm{cp}, \mathrm{ip}\rangle\}$. Thus, x 0-subjacency commands y iff for the least z such that $z > x$, $z \geq y$: the set $V := \{u : x < u < z\}$ does not contain a subsequence $\langle x_0, x_1\rangle$ of points such that $x_0 \vDash \mathrm{ip}$ and $x_1 \vDash \mathrm{cp}$ (see [7]).

We are interested in such relations $D(S)$ of the form

(42) $$x\ D(S)\ y :\Leftrightarrow y\ C(S)\ x \text{ and } y < x$$

These are the nearness relations defined in the Koster-matrix (see [2] and [4]). They can be described by programs, which we denote by $\delta(S)$. These prorgrams have the subchain property. Suppose that γ is an execution chain of $\delta(S)$ and that γ' is a subchain. By definition, no subchain of γ is an execution chain of $D(S)$, and this holds a fortiori of γ'.

Corollary 18. *Let $\Delta = \{(\delta(S_j), j) : j < n\}$ and let $\mathcal{K}$ be the class of Δ-distance restricted reentrancy structures. Then $L(\mathcal{K})$ has the finite model property and satisfiability is in 2EXPTIME.*

It follows that the α defined through $D(S)$—and these are the ones that are of linguistic interest—are preserved by passing from $\mathfrak{M}$ to the model $\mathfrak{E}$ of egregious points. Yet, I should point out that the restriction to white relations in the definitions practically means that the distance principle define distance with respect to D-structure. Or, equivalently, if we are looking for a derivational account, they encode true movement paths only for Freeze-movement. This means that we still have to find analogous results for Shortest Steps (which is the most common type of movement).

Now what if α is not a command relation or of the form $D(S)$? Then so far anything is possible. However let us mention a particular case, namely when in place of (5) we have conditions of the form

(43) $$\langle\blacktriangledown_j\rangle p \to \langle\alpha_j\rangle p$$

where α_j contains only white relations (even in tests).

Corollary 19. *Suppose that $\mathcal{K}$ is the class of Δ-distance restricted reentrancy structures defined by distance programs of the form* (43). *Then $L(\mathcal{K})$ has the finite model property and is decidable in 2EXPTIME.*

9. Shortest Steps

Now let us consider the distance principles related to Shortest Steps Movement. They are of the form

(44) $$\langle\blacktriangledown_j\rangle p \to \langle\alpha_j\rangle p$$

where the first program of the computation trace of α_j does not contain $\blacktriangledown_j$.

Definition 20. *Call α **initially white** if there are β_i, $i < n$, such that*

$$\alpha \subseteq \bigcup_{i<m} \triangledown_i;\beta_i \tag{45}$$

We start with fact that **NRS** has the finite model property. Let $X(\varphi)$ be the following set

$$X(\varphi) := \{\langle\blacktriangledown_j\rangle\nu \to \langle\alpha_j\rangle\nu : \nu \in \mathrm{At}(\varphi), j < n\} \tag{46}$$

Furthermore, let $\mathrm{At}^+(\varphi)$ the set of atoms based on $\varphi; X(\varphi)$. By the previous results there is a finite **NRS**-model

$$\langle \mathfrak{M}, \beta, x\rangle \vDash \varphi; \boxdot X(\varphi) \tag{47}$$

We may assume the frame is generated from x via the white relations. By induction we define a sequence $\hat{\blacktriangleright}^p_j$ of relations and a sequence

$$\mathfrak{M}_p := \langle M, \{\rhd_i : i < m\}, \{\blacktriangleright_j : j < n\}\rangle \tag{48}$$

Moreover, the inductive claim is that for every $\delta \in \mathrm{FL}(\varphi; X(\varphi))$:

$$\langle \mathfrak{M}_{p+1}, \beta, x\rangle \vDash \delta \quad\Leftrightarrow\quad \langle \mathfrak{M}_p, \beta, x\rangle \vDash \delta \tag{49}$$

Denote by $a_p(x)$ the φ-atom of x in $\langle \mathfrak{M}_p, \beta\rangle$. Then (49) is true if for all x: ❶: $a_{p+1}(x) = a_p(x)$ and ❷: for all $\delta = \langle\beta\rangle\nu$ a subformula of $\langle\alpha_j\rangle\nu$ or $\delta = \langle\blacktriangledown_j\rangle\nu$ (49) holds. This is the way the results is going to be proved.

From this we get that $a_{p+1}(x) = a_p(x)$, and so by induction $a_p(x) = a_0(x)$. From (49) we get that for all $x \in M$:

$$\langle \mathfrak{M}_p, \beta, x\rangle \vDash X(\varphi) \tag{50}$$

For all points x of height $\neq p$ we set $x \hat{\blacktriangleright}^{p+1}_j y$ iff $x \hat{\blacktriangleright}^p_j y$. For x of height p we do the following. Suppose that $x \hat{\blacktriangleright}^p_j y$. Two cases arise. Either $x \xrightarrow{\alpha_j}_{\mathfrak{M}_p} y$ or not. In the first case we put $x \hat{\blacktriangleright}^{p+1}_j y$. In the second case we choose a y' such that $x \xrightarrow{\alpha_j}_{\mathfrak{M}_p} y'$ and $a_p(y') = a_p(y)$ and then put $x \hat{\blacktriangleright}^p_j y'$ (and eliminate the old arc). That y' exists is seen as follows. First, we have $\langle \mathfrak{M}_p, \beta, x\rangle \vDash \langle\blacktriangledown_j\rangle a_p(y)$, where $a_p(y)$ is the atom of y in $\langle \mathfrak{M}_p, \beta\rangle$. By (50) we have $\langle \mathfrak{M}_p, \beta, x\rangle \vDash \langle\alpha_j\rangle a_p(y)$. And so there is a y' with $x \xrightarrow{\alpha_j}_{\mathfrak{M}_p} y'$ and $a_p(y') = a_p(y)$, as desired. Now using Lemma 11 we get ❶, which is (49) for all $\delta \in \mathrm{FL}(\varphi)$. Finally, we need to establish ❷, which is (49) for formulae of the form (A) $\langle\blacktriangledown_j\rangle\nu$ or (B) $\langle\beta\rangle\nu$. Case (❷A) is immediate from the definition. Case (❷B) is done by induction on the complexity of β. $\beta = \beta' \cup \beta''$ is immediate. $\beta = \chi?$ is immediate. $\beta = \beta'^*$. Then $\beta = \top? \cup \beta';\beta'^*$ and so is reduced to the cases $\top?$ and $\beta';\beta'^*$. There remains the case $\beta = \beta';\beta''$.

Now, either β' is simple or we can reduce it analogously. Using the associativity of ; and distributivity over $\cup$ we can reduce everything to the case that β' is basic and the formula has the form $\langle\beta';\beta''\rangle\nu$, which is equivalent to $\langle\beta'\rangle\langle\beta''\rangle\nu$. Now, $\langle\beta''\rangle\nu \in \mathrm{FL}(X(\varphi))$. Assume that $\beta \neq \blacktriangledown_j$. $x \overset{\beta_j}{\to}_{\mathfrak{M}_{p+1}} y$ iff $x \overset{\beta_j}{\to}_{\mathfrak{M}_p} y$ and this gives the claim together with (49) for y. If $\beta = \blacktriangledown_j$ then let y and y' be such that $x \overset{\blacktriangledown_j}{\to}_{\mathfrak{M}_p} y$ and $x \overset{\blacktriangledown_j}{\to}_{\mathfrak{M}_{p+1}} y'$. We apply the inductive hypothesis (49) for y. (Hier ist eine Luecke: wir haben $a(y) = a(y')$, aber wir brauchen $a^+(y) = a^+(y')$.)

The inductive construction is such that if x is of height n then $x \overset{\blacktriangledown_j}{\to}_{\mathfrak{M}_p} y$ implies $x \overset{\alpha_j}{\to}_{\mathfrak{M}_p} y$ for all $p \geq n$. So if q is the height of the entire tree, the model we need is $\langle\mathfrak{M}_q, \beta\rangle$.

Theorem 21. *Suppose that $\mathcal{K}$ is the class of Δ-distance restricted reentrancy structures defined by initially white distance programs. Then $L(\mathcal{K})$ has the finite model property and is decidable in 2EXPTIME.*

Proof. The finite model property and decidability follow from the previous. Now, the complexity is more subtle. Given φ we are building a model for $\varphi; X(\varphi)$, which contains 2^n formulae of length linear in $n := |\varphi|$. Given the 2EXPTIME bound for **NRS** this gives a bound of 3EXPTIME. However, rather than cascading the proof one can work out a direct proof of an analogue of Theorem 13. □

It follows that the theory of any class of structures of generative grammar constrained by Shortest Steps movement and distance regulated by command relations is decidable.

10. Naming The Egregious Points

Given that we can bound the size of a model we can now also introduce nominals that will cover the entire frame. This is done as follows. An **address** is a sequence $\mathfrak{v} = \alpha_0; b_0; \alpha_1; b_1; \cdots; b_{n-1}; \alpha_n$, where the α_i are atoms and pairwise distinct, and $b_i < i$. Call $\sigma(\varphi)$ the set of addresses. For each address $\mathfrak{v}$ we introduce a new variable $p_{\mathfrak{v}}$. These variables are contained in the set EVar.

$$\xi := \bigvee \langle p_{\mathfrak{v}} : \mathfrak{v} \in \sigma(\varphi)\rangle \tag{51}$$

Consider now the following formula A.

$$(52)\quad \begin{aligned}\Theta := & \bigwedge\langle p_{\mathfrak{v};b;\alpha} \to [\triangledown](\alpha \to p_{\mathfrak{v};b;\alpha}) : \mathfrak{v}; b; \alpha \in \sigma(\varphi)\rangle \\ & \bigwedge\langle p_{\mathfrak{v};b;\alpha} \wedge [\triangledown_i]\neg\alpha \to \bigwedge_{i<j<m} [\triangledown_j](\alpha \to \neg\xi) : \\ & \qquad \mathfrak{v}; b; \alpha \in \sigma(\varphi), i < m\rangle \\ & \bigwedge\langle p_{\mathfrak{v};b;\alpha} \to [\triangledown_i](\beta \to \neg\xi) : \beta \in \mathfrak{v}, \beta \neq \alpha\rangle \\ & \bigwedge\langle p_{\mathfrak{v}} \to [\triangledown_i](\alpha \to p_{\mathfrak{v};j;\alpha}) : \alpha \notin \mathfrak{v}, j < i < m\rangle \\ & \bigwedge\langle p_{\mathfrak{v}} \to \neg p_{\mathfrak{w}} : \mathfrak{v} \neq \mathfrak{w}\rangle\end{aligned}$$

Further,

$$(53)\quad \Xi := \bigwedge_{\alpha\in A(\varphi)} \alpha \leftrightarrow p_\alpha \wedge [\triangledown^+]\Theta$$

Lemma 22. *For every valuation β on the set* Var *such that $\langle\mathfrak{M}, \beta, x\rangle \vDash \varphi$ there is a unique extension γ defined on* $\mathrm{Var} \cup \mathrm{EVar}$ *such that $\langle\mathfrak{M}, \gamma, x\rangle \vDash \varphi; \Xi$.*

Lemma 23. *Let $\langle\mathfrak{M}, \beta, x\rangle \vDash \Xi$. Then $w \vDash p_{\mathfrak{v}} \wedge [\triangledown_{\bullet 0}]\neg p_{\mathfrak{v}}$ iff w is egregious of address $\mathfrak{v}$.*

Thus put

$$(54)\quad E(\varphi) := \bigvee\langle p_{\mathfrak{v}} \wedge [\triangledown^+]\neg p_{\mathfrak{v}} : \mathfrak{v} \in A(\varphi)\rangle$$

Then $E(\varphi)$ is true exactly at the egregious points. For an egregious point w we have a formula q_w which is true exactly at w. Let us recall how the new model was defined on the egregious points. We have $w \lhd_i v$ iff $w \leq u \prec_i v$ and $a(w) = a(u)$. Thus the fact that $w \lhd_i v$ can be expressed as

$$(55)\quad [\triangledown^*](p_v \to \langle\triangledown_i; \triangledown^*\rangle p_w)$$

11. Broadening The Scope

Now we shall generalise the theorems even further. This will allow to derive decidability even in presence of adjunction. We start again with a structure $\langle M, \{\rhd_i : i < m\}, \{\blacktriangleright_j : j < n\}\rangle$ such that $\langle M, \{\rhd_i : i < m\}\rangle$ is a finite tree. The additional relations must satisfy a few conditions. First, we assume that $\blacktriangleright_j$ as well as its converse is a partial function. This is axiomatised as follows. Put

$$(56)\quad n(p) := \neg \boxdot \neg p \wedge \boxdot(p \to [\triangledown^+]\neg p) \wedge \boxdot \bigwedge_{i<j<m} \neg(\langle\triangledown_i; \triangledown^*\rangle p \wedge \langle\triangledown_j; \triangledown^*\rangle p)$$

The desired axiom is

(57) $$d(p, q) := n(p) \wedge \neg \boxdot \neg(q \wedge \langle\blacktriangledown_j\rangle p) \to \boxdot([\triangledown_j]p \to q)$$

It says that if p is true at a single point, then the set of points seeing p through $R([\blacktriangledown_j])$ is a singleton as well. Second, we shall require that if $x\ R([\blacktriangledown_j])\ y$ then y is within a certain distance of x. This notion of distance is what we now turn to. For the purpose of the next definition notice that if α has the subchain property, so does $\alpha^{\smile}$.

Definition 24. *An* ***oval*** *is a program of the form $\alpha^{\smile};\beta$, where both α and β have the subchain property and neither contains any of the $\blacktriangleright_j$, $j < n$.*

So the desired axiom is

(58) $$o(p) := \langle\blacktriangledown_j\rangle p \to \langle\alpha^{\smile};\beta\rangle p$$

Definition 25. *Let $\mathcal{K}$ be a class of structures $\langle M, \{>_i\colon i < m\}, \{\blacktriangleright_j : j < n\}\rangle$ such that*

① *$\langle M, \{>_i\colon i < m\}\rangle$ is an m-branching tree, all relations being partial functions.*
② *All $\blacktriangleright_j$ are partial functions.*
③ *There are ovals $\alpha_j^{\smile};\beta_j$ such that if $x \blacktriangleright_j y$ then $x \xrightarrow{\alpha^{\smile};\beta} y$.*

Then $\mathcal{K}$ is a class of ***oval-expanded trees****.*

As usual, our logic will be a $i + j + 1$-modal logic. The added modalities are all definable and used for the eye only. It is crucial to understand that since we did not require of the $\blacktriangleright_j$ that they are cycle-free we cannot use the Kleene star on programs containing any black relations. It is used only on white relations.

We now proceed to a proof that the logic of a class of oval-expanded trees is decidable. As usual it will turn out that we can construct a model from the egregious points, from which a complexity bound can be derived. We start with the logic $\mathbf{Tree}_m$, which comprises axioms for $\triangledown_i$ and $>$. This logic has the finite model property and is EXPTIME complete. For $\blacktriangledown_j$ we choose the logic $\mathbf{Alt}_1$. Thus we start with

(59) $$L := \mathbf{Tree}_M \otimes \bigotimes_{j<n} \mathbf{Alt}_1$$

This logic has the finite model property and is complete with respect to finite trees. Put

(60) $$Y(\varphi) := \{d(p_{\mathfrak{v}}, p_{\mathfrak{w}}) : \mathfrak{v}, \mathfrak{w} \in \sigma(\varphi)\}; \{o(\alpha) : \alpha \in \mathrm{At}(\varphi)\}$$

Assume that φ is satisfiable in the logic of $\mathcal{K}$. Hence $Y(\varphi)$ is $\mathcal{K}$-satisfiable, since it adds instances of the axioms. A fortiori, it is L-satisfiable in a tree.

Therefore, $Y(\varphi); \Xi$ is satisfiable as well. So, assume $\mathfrak{M}$ is a tree and that

$$\langle \mathfrak{M}, \beta, w \rangle \vDash \Xi; Y(\varphi) \tag{61}$$

We let $E(\varphi)$ be the set of egregious points, and define $\rhd_i$ as before: $u \rhd_i v$ iff v is the unique egregious point such that $v \leq v' \prec_i u$ and $a(v) = a(v')$. Further, for $j < n$, define the following function $(-)^{\spadesuit j}$. It is defined on the egregious points u with a $\blacktriangleright_j$-successor. Assume u has a successor via $\blacktriangleright_j$ in $\mathfrak{M}$. Then by assumption there is a v such that $u \overset{\alpha^{\smile};\beta}{\longrightarrow} v$. Now let $v^\spadesuit$ be an egregious point below v with the same atom. Put $u \gtrdot_j v^\spadesuit$ in the new structure. By the subchain property, it will also hold that $v^\spadesuit$ is in the oval of u in the new structure. $\gtrdot_j$ is a partial function since $v \mapsto v^\spadesuit$ is. It follows that egregious points have the same atom in the new structure as they do in the old structure (by induction on the formulae). The new structure is now

$$\mathfrak{E} := \langle E(\varphi), \{\rhd_i : i < m\}, \{\gtrdot_j : j < n\}\rangle \tag{62}$$

The valuation is $\gamma(p) := \beta(p) \cap E(\varphi)$. As before, it is shown by induction on $\chi \in \mathrm{SF}(\varphi)$ that for every egregious x:

$$\langle \mathfrak{E}, \gamma, x\rangle \vDash \chi \quad \Leftrightarrow \quad \langle \mathfrak{M}, \beta, x\rangle \vDash \chi \tag{63}$$

This is by now routine. All that needs to be shown is that the converse of $\gtrdot_j$ is a partial function as well. Pick $v^\spadesuit$. We have $\langle \mathfrak{M}, \beta, v^\spadesuit\rangle \vDash p_{\mathfrak{v}}$ for a certain $\mathfrak{v} \in \sigma(\varphi)$. It follows that also $\langle \mathfrak{M}, \beta, v\rangle \vDash p_{\mathfrak{v}}$. By construction, $u \gtrdot_j v^\spadesuit$ means that u is egregious, and so $u \vDash p_{\mathfrak{w}}$ for some $\mathfrak{w}$. Since the root satisfies $d(p_{\mathfrak{v}}, p_{\mathfrak{w}})$, we have that every point u' such that u' sees a point satisfying $p_{\mathfrak{v}}$ must satisfy $p_{\mathfrak{w}}$. Thus, $u' \geq u$, and either $u' = u$ or u' is not egregious. This shows the claim.

Theorem 26. *The modal logic of a class of oval-expanded trees has the finite model property and is decidable. A model for φ has at most 2^{2^n} points, where n is the number of subformulae of φ.*

12. Conclusion

As I have outlined in a series of papers (see [3], [4], [6], [7], [9], [10] and [11]), the theories of generative grammar, starting with GB can be modelled entirely using modal logic over five basic relations. This does not mean that there is a single logic that describes them all (see [5] for an extensive discussion). It means however that the theories describe a set of grammars, and thus a set of possible logics for grammars. In this paper I have shown that—excluding economy principles—the entire class of these logic is decidable. Thus, one can effectively decide for any pair of GB/MP theories whether they generate the same structures (not strings, as this is even undecidable in the context free case).

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