

The Formal Semantics of Quantification

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Steven Cushing

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Abstract

A quantifier is a semantic operator that answers one of the questions "How many?" or "How much?" English expressions like *all, some, many, most, no, few, nearly all, exactly six, at most nine, and some but not many* all qualify as quantifiers. The semantic phenomenon of quantification is universal because of the common world in which human beings find themselves. However languages might differ in their surface grammar, it would be surprising indeed to discover a linguistic community whose members had no interest in quantities and their comparison, even if this interested included only the proverbial *one, two, many* form of "counting". Some means of expressing quantificational notions must be provided by any natural language and it is the semantic notions themselves that are accounted for in the theory developed in the dissertation.

Quantifiers constitute a single semantic category, but languages differ widely in their syntactic and morphological treatment of individual members of that category. Part I of the dissertation briefly reviews some of the surface properties of quantifiers in Latin, Greek, Zulu, English, and German, in order to provide an empirical background against which to measure the theoretical results of the analysis developed in the later Parts. Singled out for special attention is the treatment given to the existential (*some*) and null (*no*) quantifiers in Latin and Greek, the class of quantitative pronouns (*all, only*) in Zulu, and the contrast between simple (*something*) and relativized (*some N*) quantifiers in English and German. The interdefinability of some quantifiers in terms of others plus negation and the interaction of quantifiers with the other Boolean operators are also examined in some detail.

Part II critically examines some of the semantic analyses of quantifiers that have appeared in the logical and linguistic literature, including the classical quantifiers (*all, some, no*), Altham's plurality quantifiers (*many, nearly all, few, a few, all but a few*), Rescher's and Kaplan's plurality quantifiers (*most, more than m/n*), Keenan's presuppositional quantifiers

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(*only, all but, someone plus*), and others. These analyses are evaluated and generalized in a number of ways, leading to new analyses of definite descriptions, comparative constructions (*more than*), and the definite article (*the*). A generalized notion of presupposition that involves more than three truth-values is also developed.

Part III of the dissertation draws upon the results of the first two parts to develop a precise formal explication of the quantifier as a logical operator. The two basic semantic characteristics of quantifiers, the binding property and their expressability in set theory are examined in detail and formalized. An elementary quantifier is defined to be a set-theoretically expressible binding operator and a normal form for the representation of elementary quantifiers is developed, along with a standard way of representing their expressability in set theory. A quantifier is defined as an equivalence class of standard elementary quantifiers and a set-theoretic quantificational relation is associated with each quantifier.

Part IV of the dissertation further develops and applies the theory of quantificational relations that is constructed in Part III. Quantificational relations are found to contain all of the semantic information that needs to be included in the lexical entries of quantifiers and an explicit mode of representation is developed for them. They are also found to provide a theoretical basis for understanding the difference between simple and relativized quantification and the alternative interpretations of simple elementary quantifiers (*something, someone, somewhere, sometimes*). A set-theoretic basis for quantifier interdefinability is developed and is found to lead to a mathematical group structure that helps provide a link between quantifiers and the superficially different category of modalities (*necessarily, possibly, probably, may*). These results are interpreted as providing support for the syntactic-semantic parallelism between sentences and noun phrases that has been argued for in other works in the literature.

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PREFACE

This study is the result of a long-standing fascination with quantifiers and the direct outgrowth of occasional discussions with David Kaplan and Barbara Partee of a number of exploratory papers written during 1971-1972. A number of suggestions of Joseph Emonds have also been incorporated in the text. The study could not have been written without the helpful comments and criticisms of these scholars, but none of them can be held responsible in any way for any errors or misconceptions it may contain.

Part I briefly reviews some of the superficial features of quantifiers in several natural languages with the purpose of developing an intuitive feel for the phenomenon that is analyzed in depth in the later Parts. Part II critically reviews the semantic analyses of a number of quantifiers that have appeared in the logical and linguistic literature and generalizes them in various ways. In Part III a precise formal explication of the quantifier as a semantic operator is developed and in Part IV this theory is applied to the analysis of a number of phenomena of linguistic and logical interest.

As in any serious study, the number of questions raised here is at least as great as the number answered. A lot more work remains to be done to deepen our understanding of quantification and of formal semantics in general. Any suggestions or criticisms that might help to achieve this aim will be warmly welcomed.

PART I:

Quantifiers in Natural Language

CHAPTER 1: QUANTIFIERS AND THEIR SURFACE FORMS

Section 1: Quantifiers as Substantive Universals

A quantifier is a semantic operator that answers one of the questions "How many?" or "How much?" English expressions like *all, some, many, most, no, few, nearly all, exactly six, at most nine*, and *some but not many* all qualify as quantifiers. In the rest of this chapter and in the next we will present a brief impressionistic discussion of some of the surface features of quantifiers in several natural languages, with the purpose of developing some familiarity with the phenomenon to be analyzed. This will provide an empirical background against which to measure the results of the analysis we develop. The remaining three Parts will deal with quantification solely as a semantic phenomenon without regard for how it might be expressed morphologically or syntactically in specific natural languages.

Most of the data that we examine in this study will be from English, but, quite aside from Chomsky's innateness hypothesis, which claims that universal features of language constitute evidence for principles that are innate in the very structure of the human mind, there is good reason to believe that the theory we will develop will be universal in scope. The semantic phenomenon of quantification is universal because of the common world in which all human beings find themselves. What Putnam (1968) says about common nouns is equally true of quantifiers:

If Martians are such strange creatures that they have no interest in physical objects..., their language will contain no concrete nouns; but would this not be more, not less surprising on any reasonable view, than their having an interest in physical objects?

It would be difficult to imagine a natural language, that is, a language spoken by human beings on the planet Earth, that did not provide its users with some means of expressing the answers to questions of the form "How many?" or "How much?" Certainly

a way of expressing *all* and a notion of *no* or *none* are basic to life in this world. However languages might differ in their surface grammar, it would be surprising indeed to discover a linguistic community whose members had no interest in quantities and their comparison, even if this interest included only the proverbial *one, two, many* form of "counting". Some means of expressing quantificational notions must be provided by any natural language and it is the semantic notions themselves that will be accounted for in the theory we develop.

Section 2: Classical and Plurality Quantification in Latin and Greek

Quantifiers constitute a single semantic category, but languages differ widely in their syntactic and morphological treatment of individual members of that category. This variability is difficult to appreciate in a language like English, which is relatively poor in inflections, but we can get some taste of it by examining the classical and plurality quantifiers in Latin and Greek. The classical quantifiers are the existential quantifier *some*, the universal quantifier *all*, and the null quantifier *no*. They are called "classical", because they are the most extensively used quantifiers in mathematics and the most thoroughly studied by logicians. The basic plurality quantifier is *many*, which was first analyzed semantically by Altham (1971).

Both Latin and Greek treat *many* more or less as a regular adjective. Greek πολλοί, "many", is the plural of πολὺς, "much", and is declined similarly to μέγας, "great", and ἡδύς, "sweet". Both μέγας and πολὺς are second- and first-declension adjectives with the exception of the nominative, the accusative, and the vocative singular masculine and neuter, which are of the third declension. The plural, which is what interests us most, is entirely regular and declines like any second- and first-declension adjective. Latin *multi*, "many", is the plural of *multus*, much, and is entirely regular as a first- and second-declension adjective like *magnus*, "great". We see that *many* is not distinguished morphologically from other sorts of adjective in either Latin or Greek, despite its specifically quantificational semantic status.

The same is true of *all* in Latin and almost so in Greek. Latin *omnes*, "all", plural of *omnis*, "every", is a completely regular third-declension adjective of two endings, like *fortis*, "strong". Greek πᾶς, "all", is a third- and first-declension

t-stem adjective, declined specifically like the first aorist active participles, such as παιδεύσας, "having instructed". Whether the inflectional parallel between πᾶς and these participles has any semantic significance is not at all clear to me.

We have seen that there is nothing in the treatment of the plurality or universal quantifiers in Latin or Greek that distinguishes them specifically as quantifiers, rather than ordinary adjectives. This is not the case with the null quantifier. Latin *nullus*, "no", is an adjective, like *multi* and *omnes*, but it belongs to a special class of adjectives that also includes a number of other quantifier-like operators. *Nullus* is a regular first- and second-declension adjective except for special inflections in the genitive and dative singular, which it shares with *unus*, "one", *alius*, "other", *alter*, "the other (of two)", *totus*, "whole", *ullus*, "any", and *solus*, "only". All of these forms are quantificational in character and *alius* and *alter* also have a demonstrative element in their meaning. The latter fact is reflected in the fact that the demonstrative pronoun *ille*, "that", also shares the special genitive and dative singular inflections of this class of adjectives.

Latin *nullus* is from *non*, "not", and *ullus*, and the Greek form for *no* is formed in a similar way. Greek οὐδείς, "no", is derived from οὐδέ, "not even", and εἷς, "one". Its form immediately marks it as a quantifier, rather than an ordinary adjective, because εἷς, itself a quantifier, has its own set of endings.

Both Latin and Greek give special treatment to the existential quantifier. Latin uses the form *quidam*, "some", which is derived from the relative pronoun *qui*, "who", and is declined like it. The forms of the interrogative adjective *qui*, "which?", are identical to those of this relative pronoun and those of the interrogative pronoun *quis*, "who?", are similar. Greek expresses the existential quantifier by the indefinite pronoun *τις*, "some", which is identical in all its forms to the interrogative pronoun *τίς*, "who?", except for the accents.

Section 3: Quantitative Pronouns in Zulu

In the last section we saw that Latin includes *only* in a special class of quantificational adjectives that also includes *one*, *whole*, *no*, and *any*. Zulu gives a special status to *only* and *all*, by expressing them in parallel forms that are different from its treatment of other quantifiers or adjectives.

Both *all* and *only* are expressed in Zulu by what traditional grammarians have called "quantitative pronouns". *All* is expressed by using the stem *-nke* with pronominal prefixes and *only* is expressed by using the stem *-dwa* with the same set of prefixes, the only exception being that *all* does not occur in the first and second person singular. The Zulu sentences

(1) *Izisebenzi zonke zifuna umsebenzi.*

(2) *Izisebenzi zodwe zifuna umsebenzi.*

for example, are rendered in English, respectively, by the sentences

All workers want work.

Only workers want work.

The sentences in (1) and (2) differ only in that the former uses *-nke*, "all", while the latter uses *-dwa*, "only". Both forms use the pronominal prefix *zo-* to express noun-class agreement with *izisebenzi*, "workers".

In contrast, the Zulu sentences

(3) *Izisebenzi ezincane zifuna umsebenzi.*

(4) *Izisebenzi ezide zifuna umsebenzi.*

(5) *Izisebenzi ezinye zifuna umsebenzi.*

(6) *Izisebenzi eziningi zifuna umsebenzi.*

(7) *Izisebenzi ezihlanu zifuna umsebenzi.*

are rendered in English, respectively, by the sentences

Young workers want work.

Tall workers want work.

Some workers want work.

Many workers want work.

Five workers want work.

All of these sentences use the adjectival concord *ezi-*, rather than the pronominal prefix *zo-* that we saw in the quantificational

sentences (1) and (2), despite the fact that *-nye*, "some", *-ningi*, "many", *-hlanu*, "five", in (5), (6), (7), respectively, are quantifiers semantically.

The same special treatment that Zulu gives to *all* and *only* is also used for the class of quantifiers *all n*. The Zulu sentences

(8) *Izisebenzi zombili zifuna umsebenzi.*

(9) *Izisebenzi zonhlanu zifuna umsebenzi.*

for example, are rendered in English, respectively, by the sentences

(10) *Both workers want work.*

All five workers want work.

Both (8) and (9) use the pronominal prefix *zo-*, which we saw in (1) and (2), with the stems *-bili*, "two", and *-hlanu*, "five", respectively, to express, respectively, **all two* and *all five*. *Both* in (10) is a suppletive form, by which English expresses the non-occurring **all two*. A comparison of (9) with (7) reveals that the only thing that distinguishes *five* in Zulu from *all five* is the use of *ezi-* in the former and *zo-* in the latter. As we saw earlier, *ezi-* occurs with adjectives, while *zo-* occurs with the special class of quantitative pronouns, which includes *all* and *only*. *Zo-*, along with its analogs for the other noun classes, also occurs with the so-called "absolute pronoun", which expresses *I*, *you*, *it*, and their plurals.

Section 4: Simple and Relativized Quantifiers in English and German

Languages like English and German distinguish more or less sharply between simple quantification, in which a quantifier refers to "things in general", and relativized quantification, in which two specific classes or properties are referred to explicitly. Sentences like

(11) *Some things are worth studying.*

Many things can be proven.

Few things are as interesting as linguistics.

Some but not many things are well-understood.

are instances of simple quantification, and sentences like

- (12) *Some phenomena are worth studying.*
Many linguistic claims can be proven.
Few subjects are as interesting as linguistics.
Some but not many linguistic processes are well-understood.

are instances of relativized quantification, because each of the former mentions only one class or property, while each of the latter mentions two. In (11) we have the properties of being worth studying, being provable, being as interesting as linguistics, and being well-understood, but in (12) we have all of these plus the classes of phenomena, linguistic claims, subjects, and linguistic processes.

The term "relativized" was suggested to me by David Kaplan (personal communication). Stolyar (1970) uses the term "restricted quantifiers" for the quantifiers in (2) (p. 154). Either term should be comfortable for the linguist, because the phenomenon they describe is very similar to the familiar notion of the restrictive relative clause. The sentence

- (13) *All theorems can be proven.*

for example, is synonymous with the sentence

- (14) *All things which are theorems can be proven.*

The phrase *which are theorems* in (14) is a restrictive relative clause that restricts or relativizes the quantifier *all* to the set of theorems. In (13) the same task is performed by collapsing the expression *all things which are theorems* in (14) into the relativized quantification *all theorems*.

Simple occurrences of the classical quantifiers in both English and German can be collapsed into a single word. The sentences

- Some things can be proven.*
All things can be proven.
No things can be proven.

for example, can each be reformulated, respectively, as the sentences

- Something can be proven.*
Everything can be proven.
Nothing can be proven.

giving us the relationships

- (15) *some things = something*
all things = everything
no things = nothing

This collapsibility shows us that there can be simple quantification with reference less general than to the class of all things. Sentences like

- Some people have studied logic.*
All people have studied logic.
No people have studied logic.

for example, can each be reformulated, respectively, as the sentences

- Someone has studied logic.*
Everyone has studied logic.
No one has studied logic.

giving us the relationships

- (16) *some people = someone*
all people = everyone
no people = no one

No one in (16) is spelled as two words, but this is undoubtedly the result of orthographic and pronunciation factors, rather than to any semantic difference between *no* and the other two quantifiers.

Classical time and place quantifications behave in a similar way. Time sentences like

Linguistic claims are at some times false.

Theories must at all times be tested.

Unjustified claims should at no times be made.

for example, can each be reformulated, respectively, as

Linguistic claims are sometimes false.

Theories must always be tested.

Unjustified claims should never be made.

giving us the relationships

(17) at some times = sometimes

at all times = always

at no times = never

The synonymy of the place sentences

Linguistics can be studied at some places.

Linguists can be found in all places.

At no places is there a language that has been fully described.

respectively, to the sentences

Linguistics can be studied somewhere.

Linguists can be found everywhere.

Nowhere is there a language that has been fully described.

for example, gives us the similar relationships

(18) at some places = somewhere

at all places = everywhere

at no places = nowhere

for simple place quantifications.

Corresponding to (5), (6), (7), and (8), respectively, German has the following relationships:

(15') einige Dinge = etwas

alle Dinge = alles

keine Dinge = nichts

(16') einige Leute = jemand, irgendeiner

alle Leuter = jederman

keine Leuter = niemand

(17') einige Male = manchmal

alle Male = immer, allezeit

keine Male = nie, niemals, nimmer

(18') einige Plätze = irgendwo

alle Plätze = überall, allenthalben

keine Plätze = nirgends

We see that German marks the distinction between simple and relativized quantification somewhat more sharply than English does.

In most of the English relationships the expanded form is more or less predictable from the reduced form. In (15) the reduced form can be obtained simply by dropping the plural marker and coalescing the two words into one, remembering to replace *all* with its equivalent *every*. In (16) we simply replace *people* with *one* and coalesce, again replacing *all* with *every*. The relationships in (18) are just as regular. We simply replace *place* with *where* and *all* with *every* and coalesce. The form *sometimes* in (17) is the clearest of all, since it involves only coalescence, and even *always* retains a remnant of the *all* that is contained in its meaning. *Never*, in (17), is entirely suppletive, except for its initial *n*, because *ever* is quite distinct from *sometimes*.

There are clearly regularities in the German relationships, but they are of a different character from the English ones. Every reduced form in (15), for example, is identical to its unreduced form, except for the *s* and the word boundary. In contrast, although *alles*, "everything", contains *alle*, "all", and *nichts*, "nothing", undoubtedly has its origin in *nicht etwas*, "not something", the reduced forms in (15') share no common relationship with their unreduced forms of the kind in evidence in (15). Each is formed in its own way.

Similar observations hold for the other German forms. Both *jemand*, "someone", and *niemand*, "no one", in (16') contain *-mand*, a variant, perhaps, of *man*, "one", which is also present in *jederman*, "everyone", but *je* means "ever" or "each", not "some", and *nie* means "never", not "no". Both *irgendwo*, "somewhere", and *nirgends*, "nowhere", in (18') contain *irgend*, "some", and the *n* in *nirgends* undoubtedly represents *nicht*, "not", but why does the former represent *Plätze*, "place", by *-wo* and the latter by *-s*? *Allezeit*, "always", in (17') is literally "all time" from *alle* and *Zeit*, "time", but *nimmer*, "never", is clearly from *nicht immer* and so should mean "not always", rather than "not ever", and *manch-* in *manchmal*, "sometimes", is from *mancher*, which means "many", not "some". The morphological relationship between the reduced and unreduced forms in German is clearly not as simple as the corresponding relationship in English.

The main conclusion to draw from this brief analysis is that German takes the distinction between simple and relativized quantification very seriously and goes to great lengths to distinguish the two semantic phenomena in its surface grammar. Whereas the English versions of the reduced simple quantifications look very much like their unreduced forms and might, as a result, be mistaken for relativized quantifications, the corresponding confusion is much less likely in German. We might not have noticed the distinction between simple and relativized quantifiers, if we had restricted our attention to English, but in German we simply cannot miss it.

There are even some quantifiers, such as *only*, that can occur only in relativized form. We have already seen that Latin includes *only* in its special class of quantificational adjectives and that Zulu includes *only* in a class with the quantifiers *all* and *all n*, so there should be no hesitation to accept *only* as a quantifier. A sentence like

Only theorems can be proven.

is perfectly normal, but the corresponding simple quantification

**Only things can be proven.*

is semantically anomalous, at best, and the same can be said of the simple place and time quantifications

**Linguistics can be studied only at places.*

(19) **Linguistic claims are false only at times.*

There is one reading of (19), of course, that is perfectly acceptable, namely, when it is used synonymously with the sentence

Linguistic claims are false only at some times.

which contains a different quantifier. Even the seemingly acceptable simple quantification

Only people can be linguists.

is a little peculiar, however, because what it says is already included in the meaning of *linguist*.

CHAPTER 2

QUANTIFIERS AND THE BOOLEAN OPERATORS

Section 1: Negation and Interdefinability

Altham (1971) begins a discussion of plurality quantifiers with the following observations:

The negation of the proposition

(1) *Many men are lovers.*

is, quite simply

(2) *Not many men are lovers.*

(2) is equivalent to

(3) *Few men are lovers.*

(3) in turn is equivalent to

(4) *Nearly all men are not lovers.*

The negation of (3), namely,

(5) *It is not the case that few men are lovers.*

can also, more idiomatically, be written as

(6) *Not a few men are lovers.*

(6) is equivalent to (1), and also to

(7) *Not nearly all men are not lovers.*

Finally,

(8) *Not nearly all men are lovers.*

is equivalent to

(9) *Many men are not lovers.*

and also to

(10) *Not a few men are not lovers.*

From these and other examples it can be seen that *many*, *few* and *nearly all* are related as follows:

(11) *Many = not few = not nearly all not*

Not many = few = nearly not all

Not many not = few not = nearly all

Hence *few* and *nearly all* can be defined in terms of *many* and *not*. Indeed any two of *many*, *few* and *nearly all* can be defined in terms of the third and negation. (p. 1)

Altham uses these relationships to develop natural deductive systems for the three quantifiers *many*, *few*, and *nearly all*.

As Altham points out, the same kind of relationship also holds for the classical quantifiers. The sentence

Some linguists have studied logic.

is equivalent to each of the sentences

It is not the case that no linguists have studied logic.

It is not the case that all linguists have not studied logic.

The sentence

No theory is perfect.

is equivalent to each of the sentences

It is not the case that some theory is perfect.

All theories are imperfect.

The sentence

All linguists should study logic.

is equivalent to each of the sentences

It is not the case that some linguist should not study logic.

No linguist should fail to study logic.

It follows that *some*, *no*, and *all* are related by the equivalences

(12) *some* = *not no* = *not all not*

not some = *no* = *all not*

not some not = *no not* = *all*

just as *many*, *few*, and *nearly all* are related by the equivalences in (11).

The relations in (11) and (12) are reflected in the simple time and place versions of their quantifiers. Corresponding to (17) and (18) we also have the relationships

(13) *at many times* = *often*, *frequently*

at few times = *seldom*, *infrequently*, *rarely*

at nearly all times = *almost always*

As (12) would predict, the sentences

(a) *Linguistic claims are sometimes false.*

(b) *Theories must always be tested.*

(c) *Unjustified claims should never be made.*

are equivalent, respectively, to each member of the corresponding pair of sentences

(a) *It is not the case that linguistic claims are never false.*

It is not the case that linguistic claims are always true.

(b) *It is not the case that theories should sometimes not be tested.*

Theories should never be left untested.

(c) *It is not the case that unjustified claims should sometimes be made.*

Unjustified claims should always not be made.

The sentences

(a) *Linguistics can be studied somewhere.*

(b) *Linguists can be found everywhere.*

(c) *Nowhere is there a language that has been fully described.*

for example, are equivalent, respectively, to each member of the corresponding pair of sentences

(a) *It is not the case that linguistics can be studied nowhere.*

It is not the case that studying linguistics is impossible everywhere.

(b) *It is not the case that there is somewhere that linguists cannot be found.*

Nowhere is it the case that linguists cannot be found.

(c) *It is not the case that somewhere there is a language that has been fully described.*

Languages everywhere have not been fully described.

The syntactic relationships that hold among these sentences may turn out to be quite complex, but the semantic relationships are clearly those described in (12).

As (11) would predict, in conjunction with (13), similar relationships hold for the simple time versions of Altham's quantifiers. The sentences

(a) *Linguists often propose new universals.*

(b) *Linguists seldom prove their claims.*

(c) *Linguists almost always use English as their evidence.*

for example, are equivalent, respectively, to each member of the corresponding pair of sentences

- (a) *It is not the case that linguists seldom propose new universals.*

It is not the case that linguists almost always fail to propose new universals.

- (b) *It is not the case that linguists often prove their claims.*

Linguists almost always do not prove their claims.

- (c) *It is not the case that linguists often do not use English as their evidence.*

Linguists seldom fail to use English as their evidence.

English does not contain collapsed place forms for these quantifiers.

Section 2: Conjunction and Disjunction

Altham suggests that a number of English quantifiers can be analyzed semantically as the conjunctions of other quantifiers. He proposes that a *few*, for example, as opposed to *few*, should be analyzed as the conjunction of *some* and *not many*, because a sentence like

- (14) *A few people understand quantifiers.*

is synonymous with a corresponding sentence like

- (15) *Some but not many people understand quantifiers.*

Sentence (15), in turn, is synonymous with the sentence

Some people understand quantifiers but not many people understand quantifiers.

which has sentences, rather than quantifiers, as its conjuncts. Actually, Altham analyzes sentences like (14) directly in terms of sentences like (16), without talking explicitly about the conjunction of quantifiers themselves. We will see, however, that the notion of quantifier conjunction and disjunction that can be derived from his approach is interesting in its own right and has an important theoretical application.

Part II

Semantic Analyses of Specific Quantifiers

CHAPTER I

THE CLASSICAL QUANTIFIERS

Section 1: The Universal Quantifier

Semantic analyses of specific quantifiers, like those of other logical operators, are customarily given as part of a definition of logical "satisfaction". As Mendelson (1964) describes this notion,

An interpretation consists of a non-empty set D , called the domain of the interpretation, and an assignment to each predicate letter A_j^n of an n -place operation in D (i.e., a function from D^n into D), and to each individual constant a_i of some fixed element of D .

For a given interpretation, a wf without free variables (called a closed wf) represents a proposition which is true or false, whereas a wf with free variables stands for a relation on the domain of the interpretation which may be satisfied (true) for some values in the domain of the free variables and not satisfied (false) for the others.

(p. 49)

The term "wf" in this discussion stands for "well-formed formula" and denotes, roughly, the logician's equivalent of what linguists would call a "grammatical sentence" in natural language.

To construct a formal definition of satisfaction, we let there be given an interpretation with domain D and we let Σ be the set of denumerable sequences of members of D . What we want to define is

what it means for a sequence $s=(b_1, b_2, \dots)$ in Σ to satisfy a wf A under the given interpretation. As a preliminary step, we define a function s^* of one argument, with terms as arguments and values in D .

- (1) If t is x_i , let $s^*(t)$ be b_i .
- (2) If t is an individual constant, then $s^*(t)$ is the interpretation in D of this constant.

- (3) If f_i^n is a function letter and g is the corresponding operation in D , and t_1, \dots, t_n are terms, then $s*(f_i^n(t_1, \dots, t_n)) = g(s*(t_1), s*(t_2), \dots, s*(t_n))$.

Thus, s^* is a function, determined by the sequence s , from the set of terms into D . Intuitively, for a sequence $s=(b_1, b_2, \dots)$ and a term t , $s^*(t)$ is the element of D obtained by substituting, for each i , b_i for all occurrences of x_i in t , and then performing the operations of the interpretation corresponding to the function letters of t .

(p. 50)

The function s^* , in other words, is simply the familiar process of substitution. If a constant symbol occurs in a formula, then s^* maps it onto the individual whose name it is. For the i^{th} variable in a formula, s^* substitutes the i^{th} individual in s . If a function symbol occurs in a formula, then s^* interprets it as the operation it represents.

Mendelson defines satisfaction inductively as follows:

(i) if A is an atomic wf $A_i^n(t_1, \dots, t_n)$ and B_j^n is the corresponding relation of the interpretation, then the sequence satisfies A if and only if $B_j^n(s^*(t_1), \dots, s^*(t_n))$, i.e., if the n -tuple $(s^*(t_1), \dots, s^*(t_n))$ is in the relation B_j^n .

(ii) s satisfies $\neg A$ if and only if s does not satisfy A .

(iii) s satisfies $A \supset B$ if and only if either s does not satisfy A or s satisfies B .

(iv) s satisfies $(x_i)A$ if and only if every sequence of Σ which differs from s in at most the i^{th} component satisfies A .

Intuitively, a sequence $s=(b_1, b_2, \dots)$ satisfies a wf A if and only if, when we substitute, for each i , a symbol representing b_i for all free occurrences of x_i in A , the resulting proposition is true under the given interpretation.

A wf A is true (for the given interpretation) if and only if no sequence in Σ satisfies A .

A is false (for the given interpretation) if and only if no sequence in Σ satisfies A .

An interpretation is said to be a model for a set Γ of wfs if and only if every wf in Γ is true for the interpretation.

(p. 51)

Mendelson's rule (i) says that a formula that consists of a predicate symbol followed by some individual symbols is satisfied if and only if the individuals that are substituted for the individual symbols really do stand in the relation represented by the predicate symbol. Rules (ii) and (iii) are just the definition of negation and material implication, respectively. Rule (iv) says that a formula of the form $(x_i)A$ is satisfied if and only if the formula A is satisfied no matter what individual we assign as a value to x_i . Since (x_i) is Mendelson's way of writing the universal quantifier, this amounts to a semantic analysis of all and it seems to be correct:

Church (1956) gives an analysis of all that is formulated directly in terms of the truth-values truth (t) and falsity (f). Like Mendelson's, his definition is inductive:

c_0 . A wff consisting of a propositional variable a standing alone has the value t for the value t of a , and the value f for the value f of a .

c_n . Let $f(a_1, a_2, \dots, a_n)$ be a wff in which f is an n -ary function variable, and a_1, a_2, \dots, a_n are individual variables, not necessarily all different. Let b_1, b_2, \dots, b_m be the complete list of different individual variables among a_1, a_2, \dots, a_n . Consider a system of values b of f , and b_1, b_2, \dots, b_m of b_1, b_2, \dots, b_m ; and let a_1, a_2, \dots, a_n be the values which are thus given to a_1, a_2, \dots, a_n in that order. Then the value of $f(a_1, a_2, \dots, a_n)$ for the system of values b, b_1, b_2, \dots, b_m of f, b_1, b_2, \dots, b_m in that order is $b(a_1, a_2, \dots, a_n)$.

d . For a given system of values of the free variables of $\neg A$, the value of $\neg A$ is f if the value of A is t ; and the value of $\neg A$ is t if the value of A is f .

e . For a given system of values of the free variables of $[A \supset B]$, the value of $[A \supset B]$ is t if either the value of B is t or the value of A is f ; and the value of $[A \supset B]$ is f if the value of B is f and at the same time the value of A is t .

f . Let a be an individual variable and let A be any wff. For a given system of values of the free variables of $(\forall a)A$, the value of $(\forall a)A$ is t if the value of A is t for every value of a ; and the value of $(\forall a)A$ is f if the value of A is f for at least one value of a .

Rule c_0 says that a formula that consists only of a propositional variable has whatever truth-value is assigned to that variable. Rules c , d , and e are essentially the same as Mendelson's rules (i), (ii), and (iii), respectively. Rule f , like Mendelson's rule (iv), says that a universal quantification $(\forall a)A$ is true if and only if A is true no matter what individual is assigned as a value to a , but it says this more directly than Mendelson's rule does. If a single value of a makes A false, then that is enough to make $(\forall a)A$ false as well. This gives us a semantic analysis of $(\forall a)$, Church's symbolism for *all*, and it is the same analysis as that given by Mendelson.

A much more elegant analysis of *all* is given by van Fraassen (1971) in a somewhat more sophisticated notation and terminology. According to van Fraassen,

A factual situation comprises a set of individuals bearing certain relations to each other. Hence a situation can be represented by a relational structure $\langle D, R_1, \dots, R_n, \dots \rangle$, where D is the set of individuals in question and R_1, \dots, R_n, \dots certain relations on D . If we wish to describe this relational structure in a language with a quantificational syntax, we assign some member of D to each variable as its denotation, and some n -ary relation on D to each n -ary predicate as its extension. The function used to make the assignment to the predicates is called an interpretation function, and the set D a domain of discourse. Together they make up a model for the syntax. We can specify the model...by specifying a domain D and interpretation function f (pp. 107-108)

A model, in other words, is a set of individuals whose names are the variables of the language and which stand, variously, in the relations denoted by the predicates of the language.

The term *quantificational syntax* is used by van Fraassen to refer, in essence, to a language that includes the universal quantifier, but we can interpret his discussion more generally as applying to languages with other quantifiers as well. In his formal definitions, van Fraassen abbreviates this notion as *QCS*:

DEFINITION. A model for a QCS is a couple $M = \langle f, D \rangle$, where D is a nonempty set (the domain of M); f is a function (the interpretation function of M) defined for each predicate of the QCS, and such that if P^n is an n -ary predicate, then $f(P^n) \subseteq D^n$.

A mapping d of the variables of a QCS into the domain D of the model $M = \langle f, D \rangle$ for that QCS is called an assignment function (for M , or for D and for that QCS). (p. 108)

Truth in a model is defined, for van Fraassen, relative to an assignment of values to the variables. He denotes the relation d satisfies A in M by $M \models A[d]$ and he defines it inductively as follows:

DEFINITION. If $M = \langle f, D \rangle$ is a model for a QCS and d an assignment function for M , then \models is the least realization such that

- (a) $M \models (x_1 = x_2)[d]$ iff $d(x_1) = d(x_2)$;
- (b) $M \models (P^n x_1 \dots x_n)[d]$ iff $\langle d(x_1), \dots, d(x_n) \rangle \in f(P^n)$;
- (c) $M \models (A \ \& \ B)[d]$ iff $M \models A[d]$ and $M \models B[d]$;
- (d) $M \models (\neg A)[d]$ iff $\neg M \models A[d]$;
- (e) $M \models (x_1)A[d]$ iff $M \models A[d']$ for all assignments d' for M which are like d except perhaps at x_1 (symbolically, $d' = \frac{x_1}{d}$) for all sentences A , B , variables x_1, \dots, x_n , and n -ary predicates P^n of that QCS.

The symbol *iff* in this definition is an abbreviation for *if and only if*. The notation $\neg M \models A[d]$ in rule (d) is used to denote the denial of $M \models A[d]$. The notation $M \models A$, meaning A is true in M , is also used for the statement that $M \models A[d]$ for all assignment functions d for M .

Rule (a) says that a formula that states the identity of two variables is true if and only if the two variables are names of the same individual. Rule (b) is essentially the same as Mendelson's rule (i) and Church's rule c_0 . The effect of rules (c) and (d) is the same as that of Mendelson's (iii) and Church's e and d , but van Fraassen chooses to adopt conjunction as a primitive notion, rather than material implication. The choices are equivalent, because either notion can be defined in terms of the other plus negation.

Again, the last rule gives us a semantic analysis of *all*, because this is what van Fraassen means by (x_1) . Rule (e) says that a formula of the form $(x_1)A$ is satisfied if and only if the formula A is satisfied no matter what individual we assign as a value to x_1 . Since this is exactly the effect of Mendelson's rule (iv) and Church's rule f , it provides us with exactly the same semantic analysis.

Section 2: The Existential and Null Quantifiers

Neither Mendelson, Church, nor van Fraassen gives an explicit analysis of the existential quantifier *some* because of its interdefinability with *all*. In Church's words,

It should be informally clear to the reader that not both the universal and the existential quantifier are actually necessary in a formalized language, if negation is available. For it would be possible, in place of $(\exists x)$, to write always $\sim(x)\sim$; or alternatively, in place of (x) , to write always $\sim(\exists x)\sim$. And of course likewise with any other variable in place of the particular variable x .

Both the universal and existential quantifiers are actually necessary in an account of natural language, however, because both occur naturally. Interdefinability for the logician is a useful device that enables him to reduce the number of kinds of expressions that he includes in the artificial language he constructs. Interdefinability for the linguist, however, is an empirically discovered fact about the kinds of expressions that do occur, despite him, in the natural languages he studies. While a logician can choose to omit *some* from a formal language he is constructing, the linguist must include *some* in a description of a language in which it occurs.

The difference between these two perspectives on interdefinability is underscored by the case of the null quantifier. *No* is intuitively a quantifier, because it answers one of the questions *How many?* or *How much?*, and it is interdefinable with both *all* and *some*, as we saw in 1,2,1. Although it is common, however, for logicians at least to point out, like Church, that the one of *all* or *some* that they omit could be included in their formal language, I have found none who bother to do so for *no*. Church, Mendelson, and van Fraassen, for example, all state explicitly that *some* can be defined in terms of *all*. Mendelson tells us that

it was unnecessary for us to use the symbol E as a primitive symbol, because we can define existential quantification as follows

(1) $(Ex)A$ stands for $\sim((x)(\sim A))$

This definition is obviously faithful to the meaning of the quantifiers.

(p. 47)

Van Fraassen mentions in passing that "we shall henceforth use $\dots(Ex)A$ for $\sim(x)\sim A$ " (p. 102) and Church discusses interdefinability in the passage we cited earlier. None of these logicians even mentions the quantifier *no*, however, let alone give it a formal definition or analysis.

The linguist, in contrast must have some account of *no*, because *no* occurs in the data. Such an account can be formulated in terms of these other quantifiers plus negation or it can be formulated explicitly as a separate clause in the definition of satisfaction. For *some*, for example, Mendelson points out that the following rule is deducible from (1):

(2) A sequence s satisfies $(Ex)A$ if and only if there is a sequence s' which differs from s in at most the i^{th} place such that s' satisfies A .

(p. 52)

Mendelson takes (1) as his definition of *some* and then deduces (2) as a derivative property of that quantifier. We could just as well, however, take (2) as an explicit semantic analysis of *some* by including it as a further clause in the definition of satisfaction, with the result that (1) would then be a deducible property. The important point is that some correct analysis of *some* must be included in our grammar.

In the case of *no* there are also two courses open to us. We could define *no* as either *not some* or *all not*, as we saw in 1,2,1, and the one we do not choose will then be deducible. A satisfaction rule for *no*, such as

(3) A sequence satisfies $(No x)A$ if and only if there is no sequence s' which differs from s in at most the i^{th} place such that s' satisfies A .

In Mendelson's notation, will also be deducible in either case. Alternatively, we could adopt (3) as an explicit definition of *no* by including it in the definition of satisfaction. If we choose to include explicit analyses for each of *all*, *some*, and *no* then we can prove their interdefinability from these analyses.

Whichever course we adopt for *some* and *no*, we end up with the same basic semantic result. The formula $"(Some x)A"$ is true if and only if there is at least one individual that makes $"A"$ true when it is assigned as a value to $"x"$ and the formula $"(No x)A"$ is true if and only if there is no individual that makes $"A"$ true when it is assigned as a value to $"x"$. As we saw earlier, the formula $"(All x)A"$ is true if and only if $"A"$ is true no matter what individual we assign as a value to $"x"$.

Throughout this study we will, whenever possible, use the simplified notation that was just introduced in the last paragraph. English quantifiers will be used, capitalized, to represent themselves in formulas and variables will not be italicized. We will also adopt the usual convention of naming expressions in a language by including the expressions themselves in quotation marks.

Section 3: The Relativized Case

All of the analyses that we examined in the last two sections are analyses of simple quantifiers, since no restriction was placed on the individuals that could be assigned as values to variables. We can easily derive relativized analyses by adding such restrictions to those analyses.

The only difference between the members of each of the following pairs of sentences, for example, is that the (a) sentence of each pair involves a simple quantifier, while the (b) sentence involves the corresponding relativized quantifier:

- (4) a. *Some things can be proven.*
b. *Some theorems can be proven.*
- (5) a. *Everything can be proven.*
b. *Every theorem can be proven.*
- (6) a. *Nothing can be proven.*
b. *No theorem can be proven.*

Sentence (5b) could just as well be

All theorems can be proven.

as far as our purposes are concerned. We can represent the (a) sentences in our simplified version of Mendelson's, Church's, and van Fraassen's notation by rewriting them as the formulas

- (4a') (Some x) Provable (x)
- (5a') (All x) Provable (x)
- (6a') (No x) Provable (x)

respectively. Formula (4a') is true if and only if there is at least one individual that makes "Provable(x)" true when it is

assigned as a value to " x ", formula (6a') is true if and only if there is no such individual, and formula (5a') is true if and only if "Provable(x)" is true no matter what individual we assign as a value to " x ".

Semantic representations for the (b) sentences in each pair can be constructed in an analogous way, as follows:

- (4b') (Some x) (Theorem(x), Provable(x))
- (5b') (All x) (Theorem(x), Provable(x))
- (6b') (No x) (Theorem(x), Provable(x))

respectively. The only difference between the semantic representations of the (a) sentences and those of the (b) sentences is the appearance of the extra predicate "Theorem(x)". The significance of the extra predicate in these formulas is that it places a restriction on the set of individuals that are to be considered in the quantification of the other predicate. An individual is considered in the quantification of the second predicate only if it makes the first predicate true, when assigned as a value to " x ". Formula (5b') will be true, for example, even if there is an assignment function that fails to satisfy "Provable(x)", as long as that assignment function fails to satisfy "Theorem(x)" as well. Formula (5a') will be false in such a case, however, whether or not the assignment function satisfies "Theorem(x)".

It follows that we can turn Mendelson's, Church's, and van Fraassen's analyses of simple *all* into analyses of relativized *all* in a very simple way. In Mendelson's rule (iv) we restrict the sequences of Σ , in Church's rule f we restrict the values of \underline{a} , and in van Fraassen's rule (e) we restrict the assignments \underline{d} . This gives us the following semantic analysis of relativized *all* in the different notational frameworks of these three logicians:

Mendelson: s satisfies $(x_i)(B, A)$ if and only if every sequence of Σ which differs from s in at most the i th component and which satisfies B satisfies A .

Church: Let \underline{a} be an individual variable and let A, B be any wffs. For a given system of values of the free variables of $(\forall \underline{a})(B, A)$, the value of $(\forall \underline{a})(B, A)$ is t if the value of \underline{A} is t for every value of \underline{a} for which the value of \underline{B} is t ; and the value of $(\forall \underline{a})(B, A)$ is f if the value of \underline{A} is f for at least one value of \underline{a} for which the value of \underline{B} is t .

van Fraassen: $\underline{M} \models (x_1)(B, A)[d]$ iff $\underline{M} \models A[d']$ for all assignments d' for \underline{M} which are like d except perhaps at x_1 and for which $\underline{M} \models B[d']$.

Each of these formulations is just a different way of saying that the formula " $(\forall x)(B, A)$ " is true if and only if " A " is true no matter what individual we assign as a value to " x ", as long as that individual also makes " B " true under that assignment.

Now we can either define relativized *some* and *no* in terms of relativized *all*, as we did in the simple case, or we can construct explicit semantic analyses for them analogous to (2) and (3), respectively. Since the first predicate serves simply as a restriction, the negations operate only on the second predicate, so we get

$$(\text{Some } x)(B, A) = \neg(\forall x)(B, \neg A)$$

$$(\text{No } x)(B, A) = (\forall x)(B, \neg A)$$

as the interdefinability relationships.

To construct explicit semantic analyses we proceed as follows. Formula (4b') is true if and only if there is at least one individual that makes " $\text{Theorem}(x)$ " true when assigned as a value to " x " that makes " $\text{Provable}(x)$ " true under the same assignment. Formula (6b') is true if and only if there is no such individual. We can generalize this by adding exactly the same restriction to (2) and (3) that we added to Mendelson's rule for simple *all*. This gives us the following semantic analyses for relativized *some* and *no*, respectively:

- (7) A sequence s satisfies $(\exists x_1)(B, A)$ if and only if there is a sequence s' which differs from s in at most the i^{th} place and which satisfies B such that s' satisfies A .
- (8) A sequence s satisfies $(\text{No } x_1)(B, A)$ if and only if there is no sequence s' which differs from s in at most the i^{th} place and which satisfies B such that s' satisfies A .

Rule (7) says that " $(\text{Some } x)(B, A)$ " is true if and only if there is at least one individual that makes " B " true when assigned as a value to " x " that makes " A " true under that assignment and rule (8) says that " $(\text{No } x)(B, A)$ " is true if and only if there is no such individual.

Section 4: Reducibility of the Relativized Quantifiers to the Simple Case

We have seen that the explicit semantic analyses for the relativized classical quantifiers can be constructed from the semantic analyses of the corresponding simple quantifiers by requiring that the sequences or assignment functions that are relevant to the quantification of the second predicate must also satisfy the first predicate. This relationship between the simple and relativized versions of the classical quantifiers enables us to reduce the latter to the former by constructing, for any given relativized quantification, a simple quantification that is equivalent to it.

Each of the sentences

(9) *All logicians study quantifiers.*

(10) *All linguists study quantifiers.*

for example, involves two predicates and can be represented semantically as the respective member of the pair of formulas

(11) $(\forall x)(\text{Logician}(x), \text{Study-quantifiers}(x))$

(12) $(\forall x)(\text{Linguist}(x), \text{Study-quantifiers}(x)).$

The symbols "*logician*" and "*Linguist*" in these formulas denote, respectively, the properties of being a logician and being a linguist and the symbol "*Study-quantifiers*" denotes the property of being one who studies quantifiers. Each of these symbols is undoubtedly further analyzable in a complete grammar, but for our purposes we can treat them as primitive symbols.

The sentences (9) and (10) can be reformulated, respectively, as the equivalent sentences

(13) *Everyone who is a logician studies quantifiers.*

(14) *Everyone who is a linguist studies quantifiers.*

These, in turn, are equivalent, respectively, to the sentences

(15) *Everyone is such that if he is a logician then he studies quantifiers.*

(16) *Everyone is such that if he is a linguist then he studies quantifiers.*

Since (15) and (16) are simple quantifications, representable semantically as

$$(17) \quad (\forall x)(\text{Logician}(x) \supset \text{Study-quantifiers}(x))$$

$$(18) \quad (\forall x)(\text{Linguist}(x) \supset \text{Study-quantifiers}(x)),$$

respectively, and since (15) and (16) are equivalent, respectively, to (13) and (14) and thus, respectively, to (9) and (10), it follows that (17) and (18) are equivalent, respectively, to (11) and (12), the semantic representations, respectively, of (9) and (10). We have reduced the relativized quantifications to simple quantifications that are logically equivalent to them.

The same procedure can be used to transform any relativized universal quantification

$$(19) \quad (\forall x)(B, A)$$

into the equivalent simple quantification

$$(20) \quad (\forall x)(B \supset A)$$

by replacing the ordered pair of predicates

$$(21) \quad (B, A)$$

with the single predicate

$$(22) \quad B \supset A$$

In fact, A and B can be any well-formed formulas (wff) in the language. As we saw on page 17, a wff can be either open, that is, with free variables, or closed, that is, without free variables, and its truth is relative to an assignment of individuals as values to variables. Replacing (21) in (19) with (22) is really just another way of restricting the assignment functions which are involved in determining the truth-value of

$$(23) \quad (\forall x) A$$

to those which satisfy B.

In Section 1 we saw that (23) is true under an assignment function \underline{d} if and only if every assignment function \underline{d}' that differs from \underline{d} only in the value it assigns to 'x' satisfies 'A'. In Section 3 we saw that (19) is true under \underline{d} if and only if every \underline{d}' that satisfies 'B' satisfies 'A'. This is the same,

intuitively, as saying that, in determining the truth-value of (19) under \underline{d} , we look only at the functions \underline{d}' that satisfy 'B', not caring what happens under the functions \underline{d}' that satisfy '-B'. This means that, for (19) to be true, every \underline{d}' must satisfy either '-B', in which case we do not care what it does to 'A', or 'A'. It follows that (19) is true under \underline{d} if and only if every \underline{d}' satisfies

$$(24) \quad -B \vee A.$$

Since (24) is equivalent to (22), however, it turns out that (19) is true under \underline{d} if and only if every \underline{d}' satisfies (22), that is, if and only if (20) is true under \underline{d} . It follows that (19) is logically equivalent to (20).

A similar argument can be made for the existential and null cases. We know that

$$(25) \quad (\text{Some } x)(B, A)$$

is true under \underline{d} if and only if there is a \underline{d}' that satisfies 'B' which satisfies 'A'. This means that (25) is true if and only if there is a \underline{d}' that satisfies both 'B' and 'A'. It follows that (25) is true under \underline{d} if and only if

$$(26) \quad (\text{Some } x)(B \wedge A)$$

is true under \underline{d} . Since

$$(27) \quad (\text{No } x)(B, A)$$

is true under \underline{d} if and only if there is no \underline{d}' that satisfies 'B' which satisfies 'A', we see that (27) is equivalent to

$$(\text{No } x)(B \wedge A).$$

analogously to the equivalence of (25) and (26).

In each case we have reduced a relativized quantification to a simple quantification by replacing the ordered pair of wffs (21) with a single wff. Whereas in the universal case we replaced (21) with (22), however, the existential and null cases required that it be replaced with the conjunction

$$B \wedge A$$

instead. This difference between the universal case, on the one hand, and the existential and null cases, on the other, is simply a reflection of the interdefinability relationships that hold among *all*, *some*, and *no* and of similar relationships that hold among ' \supset ', ' \vee ', and ' \wedge '.

CHAPTER 2: PLURALITY QUANTIFIERS

Section 1: Altham's Quantifiers

1.1: The Principal Quantifiers

Altham gives semantic analyses for a plurality of quantifiers, all of which are based on an intuitive notion of "manifold". A manifold, according to Altham,

is simply a set that contains many members. In any context, a certain number n is fixed upon as being the least number of objects that can form a manifold. Having done that, a set is defined to be a manifold if and only if it contains at least n members. Within certain limits, the number n is arbitrary, but in any context n must be greater than one, since it is certain that there is no context in which one would say that there were many F s if there were only one F .

(p. 8)

A manifold, then, is simply the smallest set that we would consider as containing many members in a given context. The number n is the smallest number that we would consider as *many* in that context. Manifolds themselves play no role in Altham's formalism, but the number n is central to his analyses.

Altham points out that *many* and *nearly all* are interdefinable, as we saw in 1,2,1, but he still gives explicit semantic analyses for both of them. As is usual in the logical literature, his examples are relativized, but his analyses are simple, and these are formulated as part of a definition of satisfaction. For Altham,

An interpretation is a non-empty set D and a function that assigns to every term an element of D , to every predicate letter of degree n a set of n -tuples belonging to D , and to every propositional variable one of the truth-values, T or F .

(p. 40)

A term, in Altham's system, is simply a proper name or constant symbol. If t is a term and I is an interpretation, then an interpretation I' is said to be a t -variant of I if I' is like I except at most in what it assigns to t . This relationship between I and I' is similar to the relationship between d and d' in van Fraassen's system. Like Mendelson, Church, and van Fraassen, Altham defines satisfaction inductively, first for

propositional variables, then for atomic sentences, and then for identity, negation, conjunction, disjunction, material implication, and the four quantifiers *all*, *some*, *nearly all*, and *many*.

The rules for *nearly all*, which Altham denotes by ' N ', and *many*, which he denotes by ' M ', are formulated by him, respectively, as follows:

- (28) If A is $(Nv)B(v)$, then I satisfies A iff in every n distinct t -variants of I there is at least one which satisfies $B(t)$.

If A is $(Mv)B(v)$, then I satisfies A iff there are n distinct t -variants of I , every one of which satisfies $B(t)$.

(p. 40)

Rule (28) says that "(Nearly all x) A " is true if and only if every manifold of individuals contains at least one member that makes " A " true when assigned as a value to " x ". Rule (29) says that "(Many x) A " is true if and only if there is a manifold of individuals every one of whose members makes " A " true when assigned as a value to " x ".

Because of its interdefinability with *many* and *nearly all* Altham does not give an analysis of *few*. Such an analysis can be formulated within his framework in two ways, depending on which interdefinability relationship we choose to base it on. Defining *few* as *not many* gives us the rule

- (30) If A is $(Few v)B(v)$, then I satisfies A iff there are not n distinct t -variants of I , every one of which satisfies $B(t)$.

and defining it as *nearly all not* gives us the rule

- (31) If A is $(Few v)B(v)$, then I satisfies A iff in every n distinct t -variants of I there is at least one which does not satisfy $B(t)$.

The two rules can easily be shown to be equivalent and all of the interdefinability relationships we discussed in 1,2,1 can be derived either from (28), (29), and (30) or from (28), (29), and (31).

1.2: The Secondary Quantifiers

Altham discusses a number of other plurality quantifiers, most of which he defines in terms of his three principal ones.

First, however, he discusses two families of quantifiers that are independent of the notion of manifold. Altham points out that *many*, according to (29), is identical, for any particular choice of \underline{n} , to the numerical quantifier *at least \underline{n}* . If \underline{n} is the *least* number of things that we would consider to constitute many things in a given context, then two sentences like

Many things can be proven.

At least \underline{n} things can be proven.

mean exactly the same thing. Altham denotes the quantifier *at least \underline{n}* by the symbol " $\exists_{\underline{n}}x$ " and defines the family of such quantifiers as follows:

$$(\exists_{\underline{1}}x)Fx = (\exists x)Fx$$

$$(\exists_{\underline{n}}x)Fx = (\exists x)(Fx \ \& \ (\exists_{\underline{n-1}}y)(Fy \ \& \ y \neq x)).$$

As an analysis of the related family of numerically definite quantifiers *exactly \underline{n}* Altham also gives the recursive definition

$$(\exists_{\underline{0}}x)Fx = \sim(\exists x)Fx$$

$$(\exists_{\underline{n}}x)Fx = (\exists x)(Fx \ \& \ (\exists_{\underline{n-1}}y)(Fy \ \& \ y \neq x)).$$

He neglects to mention it, but the y in these formulas must be a variable that does not occur free in Fx .

The first quantifier that Altham defines in terms of his manifold quantifiers is a *few*. He begins by pointing out that, although a *few* is superficially similar to *few*, it is semantically a very distinct quantifier:

In Chapter 1 we distinguished *few* from a *few*. *There are a few Fs* was said to be equivalent to *There are not many Fs*, and hence to follow from *There are no Fs*. No separate treatment of *few* was therefore necessary. It was taken to be covered by the treatment of *many* together with *not*.... But *There are a few Fs* is different. So far from following from *There are no Fs* is it that it is incompatible with the latter, and actually entails *There is at least one F*. We also briefly distinguished *Nearly everything is F* from *All but a few things are F*. While *Nearly everything is F* follows from *All but a few things are F*, *All but a few things are F* actually entails that *not* everything is F. The problem now is how to treat a *few* and *all but a few*, since this is not obvious from what has been said so far.

(p. 64)

Altham suggests two methods for solving this problem. First he proposes that a *few* be defined as *some but not many*, as we noted in 1,2,2, and that *all but a few* be defined as *nearly all but not all*. Formally, this gives us the relation

$$(32) \quad (Fv)A(v) = (\exists v)A(v) \ \& \ \sim(Mv)A(v)$$

or, equivalently, the relation

$$(33) \quad (Fv)A(v) = (\exists v)A(v) \ \& \ (Few \ v)A(v)$$

as a definition of a *few*, denoted by " \underline{F} ", and the relation

$$(34) \quad (\underline{\Delta}v)A(v) = (Nv)A(v) \ \& \ \sim(\forall v)A(v)$$

as a definition of *all but a few*, denoted by " $\underline{\Delta}$ ".

Altham points out that, with these definitions, a *few all but a few* are interdefinable through negation. Just as we can define *all* as *no not* and *no* as *all not*, we can also define a *few* as *all but a few not* and *all but a few* as a *few not*. He points out further, however, that, although (32) and (34) are convenient, they

may be thought unsatisfactory, since by this analysis, if there is exactly one \underline{F} , it follows that there are a few \underline{Fs} . And it may be thought that for there to be a few \underline{Fs} there must be more than one. According to this line of thought a *few* would be a quantifier lying somewhere between *many* and *some*.

(pp. 64-65)

To "do justice to this idea" Altham invites us to take the analysis of a *few* embodied in (32), (33), and (34) as actually an account of the quantifier *at least one and at most a few*.

Altham proposes a second method of analyzing a *few*, now denoted by " \underline{f} ", and he is quick to point out that his proposed analysis of \underline{f} actually provides an account of the quantifier *at least a few*. An analysis of *exactly a few* can then be obtained as the conjunction of \underline{F} and \underline{f} , giving us " $(fv)A(v) \ \& \ (Fv)A(v)$ " or, equivalently, " $(fv)A(v) \ \& \ \sim(Mv)A(v)$ " as an analysis of *There are exactly a few individuals that have \underline{A}* .

To define the semantics of \underline{f} itself Altham gives us the explicit analysis

$$(35) \quad \text{An interpretation } I \text{ satisfies } (fv)A(v) \text{ iff there are at least } \underline{m} \text{ distinct } \underline{t}\text{-variants of } \underline{I}, \text{ all of which satisfy } \underline{A}(\underline{t}).$$

(p. 65)

where m is a positive integer strictly less than n , the number chosen to define *many*, and strictly greater than one. As Altham points out, this method of defining a *few* works only if n is greater than two. What (35) says is that to say that a *few* things have a certain property is to say that there is a set of m objects, all of which have that property. The fact that a *few* is being defined, rather than *many*, is accounted for by the restriction on the size of m . In all other respects, that is, other than the fact that m is mentioned, rather than n , (35) is identical to the analysis of *many* embodied in (29).

Once he has defined "f", Altham then goes ahead and uses it to define two more quantifiers. He denotes *very nearly all* by V and defines it by the formula

$$(36) (Vv)A(v) = \sim(fv)\sim A(v)$$

pointing out that it could also be defined explicitly by the rule

$$(37) \underline{I} \text{ satisfies } (Vv)A(v) \text{ iff in every } m \text{ distinct } \underline{t}\text{-variants of } \underline{I}, \text{ at least one satisfies } A(\underline{t}).$$

Except for the fact that (37) mentions m instead of n , the rule is identical to (28), just as (35) is identical to (29) with the same proviso. Finally, Altham suggests that *all but at most a very few* might be defined in terms of "V" as

$$(38) (Vv)A(v) \ \& \ (\exists v)\sim A(v),$$

analogously to his definition of " \underline{I} " in terms of "f".

Section 2: Rescher's and Kaplan's Quantifiers

Kaplan (1966a) gives an explicit semantic analysis of the plurality quantifier *most*, based on the discussion of *most* in Rescher (1962). Kaplan includes the following clause in the definition of satisfaction:

$$(39) \text{ f satisfies } M_\alpha \phi \text{ in } \langle DR \rangle \text{ if and only if} \\ K(E_x[x \in D \text{ and } f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle]) \\ > K(E_x[x \in D \text{ and } f_x^\alpha \text{ satisfies } \neg \phi \text{ in } \langle DR \rangle]).$$

As for van Fraassen, the "D" in (39) stands for the domain of the model, a non-empty set of individuals. "R", like van Fraassen's "f", is an assignment of denotations, that is, individuals as values, to non-logical constants. Van Fraassen's "f" assigns denotations only to predicates only because his language has no

individual constants. The "f" in (39) is like van Fraassen's "d", an assignment of individuals in D as values to variables. The symbol "K" denotes the set-theoretic function *cardinality*, which maps any set onto the number (finite or otherwise) of its members. The symbol " E_x " denotes the extension of the formula that appears in its scope, that is, the set of individuals that make the formula true, when they are assigned individually as values to "x". The symbol " f_x^α " denotes the assignment of values to variables that is identical to f except that it assigns the individual denoted by "x" to the variable denoted by " α ". The symbol " α " itself is a variable that takes variables as values. In (39) "x" takes on all individuals in D as values, independently of f . Rule (39) says that the formula " $M_\alpha \phi$ " is true if and only if there are more individuals in D that make " ϕ " true than that make it false, when assigned individually as values to the variable in " ϕ " that is denoted in " $M_\alpha \phi$ " by " α ". Since " M " is Kaplan's way of writing *most*, this provides us with a semantic analysis of that quantifier.

Kaplan's logical framework is closer to that of van Fraassen than to that of any of the other logicians we have considered, but, as we just saw, his specific notation differs from van Fraassen's. In III we will see that Kaplan's notation and framework are the best-suited to the development of a general theory of quantification and, in fact, follow naturally from its underlying semantic structure.

Kaplan (1966b) generalizes (39) to get an analysis of the family of quantifiers *more than m/n*. Rather than simply adding a clause to the definition of satisfaction, however, he introduces the new notion of " m/n -satisfaction", which he defines as follows:

$$(40) \text{ f } m/n\text{-satisfies } M_\alpha \phi \text{ in } \langle DR \rangle \text{ if and only if} \\ n \cdot K(E_x[x \in D \wedge f_x^\alpha \text{ } m/n\text{-satisfies } \phi \text{ in } \langle DR \rangle]) \\ > m \cdot K(D).$$

As Kaplan points out, (40) provides a family of reinterpretations of his and Rescher's plurality quantifier symbol " M ". These reinterpretations constitute the family of quantifiers *more than m/n*, with the symbol " M " interpreted more generally than in (39). He notes that the interpretation of " M " in (39) is equivalent to the special case of (40) in which m/n is taken to be $1/2$ and that the existential quantifier is the special case in which $(m/n)=0$. Because of this systematically ambiguous use of " M " in Kaplan's rule, we will use the symbol "*Most*", rather than " M ", from now on in formulas that represent *most*.

Kaplan neglects to point out that (40) is circular, unless we specify that m/n -satisfaction is to be taken to be ordinary

satisfaction for formulas that contain no occurrences of "M" (or our "Most"). We can reformulate (40) as an explicit analysis of the family of quantifiers *more than m/n*, without the ambiguous symbol "M" and the special notion of satisfaction that interprets it, by introducing "More than m/n" as a quantifier symbol for each value of m/n and including the clause

$$(41) \quad f \text{ satisfies More than } m/n_{\alpha} \phi \text{ in } \langle DR \rangle \text{ if and only if} \\ n \cdot K(E[x \in D \wedge f_x^{\alpha} \text{ satisfies } \phi \text{ in } \langle DR \rangle]) \\ > m \cdot K(D)$$

in the definition of satisfaction. Rule (41) says that the formula "More than m/n_α φ" is true if and only if more than m/n of the individuals in *D* make φ true when they are assigned individually as values to the variable denoted by α. This gives us an explicit semantic analysis of the quantifier *more than m/n*

Section 3: The Relativized Case

3.1 Plurality Quantifiers that Do Not Involve Manifolds

Analyses of the relativized versions of Altham's numerical quantifiers and of Rescher's and Kaplan's quantifiers are easy to construct. Since the numerical quantifiers are defined recursively in terms of *some*, we simply replace simple *some* throughout these definitions with relativized *some*, which we have already analyzed in 11,1,3. This gives us

$$(\exists_1 x)(Gx, Fx) = (\exists x)(Gx, Fx) \\ (\exists_n x)(Gx, Fx) = (\exists x)(Gx, (Fx \& (\exists_{n-1} y)(Gy, (Fy \& y \neq x))))$$

as an analysis of relativized *at least n* and

$$(\exists_0 x)(Gx, Fx) = \sim(\exists x)(Gx, Fx) \\ (\exists_n x)(Gx, Fx) = (\exists x)(Gx, (Fx \& (\exists_{n-1} y)(Gy, (Fy \& y \neq x))))$$

as an analysis of relativized *exactly n*. The only difference between these analyses and the simple analyses that we saw in Section 1.2 is that individuals are required to belong to the extension of "Gx" before they can be considered in the quantification of "Fx". This requirement follows from our analysis of relativized *some* in 11,1,3 and is exactly the intuitive content of relativization, which we examined in 1,1,4. As in the simple case, "y" must be taken to be a variable that does not occur free in "Fx" or, this time, in "Gx".

The relativization requirement that individuals must be in the extension of the first wff in order to be considered in the quantification of the second wff is particularly easy to state in the case of Rescher's and Kaplan's quantifiers, because our analyses of the corresponding simple quantifiers are already formulated in terms of extensions. All we have to do is guarantee that only assignments of individuals as values to variables that satisfy the first wff are considered in the quantification of the second wff. This gives us

$$(42) \quad f \text{ satisfies Most}_{\alpha}(\psi, \phi) \text{ in } \langle DR \rangle \text{ if and only if} \\ K(E[x \in D \text{ and } f_x^{\alpha} \text{ satisfies } \psi \text{ in } \langle DR \rangle \text{ and } f_x^{\alpha} \\ \text{satisfies } \phi \text{ in } \langle DR \rangle]) \\ > K(E[x \in D \text{ and } f_x^{\alpha} \text{ satisfies } \psi \text{ in } \langle DR \rangle \text{ and } \\ f_x^{\alpha} \text{ satisfies } \neg \phi \text{ in } \langle DR \rangle])$$

and

$$(43) \quad f \text{ satisfies More than } m/n_{\alpha}(\psi, \phi) \text{ in } \langle DR \rangle \text{ if} \\ \text{and only if} \\ n \cdot K(E[x \in D \wedge f_x^{\alpha} \text{ satisfies } \psi \text{ in } \langle DR \rangle \wedge \\ f_x^{\alpha} \text{ satisfies } \phi \text{ in } \langle DR \rangle]) \\ > K(E[x \in D \text{ and } f_x^{\alpha} \text{ satisfies } \psi \text{ in } \langle DR \rangle])$$

as analyses, respectively, of relativized *most* and *more than m/n*. Whereas (39) gives us the meaning of "Most things are φ's" and (41) gives us the meaning of "More than m/n of all things are φ's", (42) gives us the meaning of "Most ψ's are φ's" and (43) gives us the meaning of "More than m/n of all ψ's are φ's". This is exactly what we expect of relativization.

3.2: Plurality Quantifiers that Do Involve Manifolds

The relativization of most of Altham's quantifiers requires a little more thought than that of his numerical quantifiers because of their crucial dependence on the manifold size index *n*. The numerical quantifiers *at least n* and *exactly n* both explicitly contain "*n*", so the appearance of "*n*" in their semantic analyses or recursive definitions poses no problem. Different choices of *n* produce different quantifiers and that is all there is to it. *Many*, in contrast, along with the various quantifiers that can be defined in terms of it, is systematically ambiguous with respect to *n*. For any particular choice of *n* *many* is identical to the specific quantifier *at least n* and which such quantifier it is identical to depends on our choice of *n*. No matter which value of *n* we choose, however, *many* remains the same quantifier *many*. This systematic ambiguity is precisely what distinguishes the single quantifier *many* from the family of quantifiers *at least n*.

Some implications of this systematic ambiguity become apparent, when we examine sentences like

(44) *Many linguists study hieroglyphics.*

(45) *Many Coptic scholars study hieroglyphics.*

We would like to have a semantic analysis of relativized *many* that makes the formulas

(46) $(\text{Many } x)(\text{Linguist}(x), \text{Study-hieroglyphics}(x))$

(47) $(\text{Many } x)(\text{Study-Coptic}(x), \text{Study-hieroglyphics}(x))$

the semantic representations, respectively, of (44) and (45). Suppose, however, that we try to relativize our analysis (29) of simple *many* in the same way that we relativized our other simple quantifiers. All we have to do, according to this procedure, is to add a statement to (29) that restricts the individuals that are considered in the quantification of " $B(t)$ " to the extension of another wff. This would transform (29) into the rule

(48) If A is $(\text{Mv})(C(v), B(v))$, then I satisfies A iff there are n distinct t -variants of I that satisfy $C(t)$, every one of which satisfies $B(t)$

as a purported semantic analysis of relativized *many*.

Suppose now that we try to use (48) to determine the truth-values of (46) and (47), the semantic representations of (44) and (45), respectively. Rule (48) says that (46), and thus (44), is true if and only if there are (at least) n distinct linguists who study hieroglyphics, while (47), and thus (45), is true if and only if there are some n distinct Coptic scholars who study hieroglyphics, for some number n whose value has been determined independently of (46) and (47). Such a statement is clearly false, however, for (44) and (45). Since there are very few Coptic scholars in the world, we might argue that (45) is true if we could find even twenty such people who study hieroglyphics. We would say that *some* linguists study hieroglyphics and that a *few* linguists study hieroglyphics, but we would not say that *many* linguists study hieroglyphics. Since there are substantially more linguists than there are Coptic scholars, the minimal number of linguists that we would consider to be many is substantially larger than the minimal number of Coptic scholars that we would consider to be many. The problem with (48) is its inherent assumption that the same choice of n will do for both.

We can remedy this defect in (48) very easily by writing the manifold size index as a function of the first wff. This gives us the rule

(49) If A is $(\text{Mv})(C(v), B(v))$, then I satisfies A iff there are $n(C)$ distinct t -variants of I that satisfy $C(t)$, every one of which satisfies $B(t)$

as our semantic analysis of relativized *many*. Rules (28) and (30) or (31) can be relativized in exactly the same way, giving us

(50) If A is $(\text{Nv})(C(v), B(v))$, then I satisfies A iff in every $n(C)$ distinct t -variants of I there is at least one that satisfies $C(t)$ which satisfies $B(t)$

as our semantic analysis of relativized *nearly all* and either

(51) If A is $(\text{Few } v)(C(v), B(v))$, then I satisfies A iff there are not $n(C)$ distinct t -variants of I that satisfy $C(t)$, every one of which satisfies $B(t)$

or

(52) If A is $(\text{Few } v)(C(v), B(v))$, then I satisfies A iff in every $n(C)$ distinct t -variants of I that satisfy $C(t)$ there is at least one which does not satisfy $B(t)$

as our semantic analysis of relativized *few*.

With a little hindsight we can see that Altham's analyses (28) and (29) and our analyses (30) and (31) suffer from an adequacy very similar to that of (48). Simple quantifications like

(53) *Many people are linguists.*

(54) *Many people are Coptic scholars.*

are really equivalent to existential statements of a certain kind. Sentence (53) means the same thing as

(55) *There are many linguists.*

and sentence (54) means the same thing as

(56) *There are many Coptic scholars.*

just as the simple quantifications

Some people are linguists (Someone is a linguist)

Some people are Coptic scholars (Someone is a Coptic scholar)

mean the same thing, respectively, as the existential statements

There are linguists.

There are Coptic scholars.

The number of Coptic scholars that there would have to be to make us agree that (56) is true, however, is substantially smaller than the number of linguists that there would have to be to make us agree that (55) is true. As we saw in the case of (44) and (45), the minimal number of linguists that constitute a manifold is substantially larger than the minimal number of Coptic scholars that constitute a manifold. The manifold size index is a function of the wff in the simple case, as well as in the relativized case, of *many*.

To accomodate the functional character of \underline{n} , we must replace (28) and (29), respectively, with the new formulations

(57) If \underline{A} is $(\underline{Nv})\underline{B}(v)$, then \underline{I} satisfies \underline{A} iff in every $\underline{n}(\underline{B})$ distinct \underline{t} -variants of \underline{I} there is at least one which satisfies $\underline{B}(\underline{t})$

(58) If \underline{A} is $(\underline{Mv})\underline{B}(v)$, then \underline{I} satisfies \underline{A} iff there are $\underline{n}(\underline{B})$ distinct \underline{t} -variants of \underline{I} , every one of which satisfies $\underline{B}(\underline{t})$

as our semantic analyses of *nearly all* and *many*, respectively, and we must replace (30) and (31), respectively, with the new formulations

(59) If \underline{A} is $(\underline{Few v})\underline{B}(v)$, then \underline{I} satisfies \underline{A} iff there are not $\underline{n}(\underline{B})$ distinct \underline{t} -variants of \underline{I} , every one of which satisfies $\underline{B}(\underline{t})$

(60) If \underline{A} is $(\underline{Few v})\underline{B}(v)$, then \underline{I} satisfies \underline{A} iff in every $\underline{n}(\underline{B})$ distinct \underline{t} -variants of \underline{I} there is at least one that does not satisfy $\underline{B}(\underline{t})$

as our alternative analyses of simple *few*. In what follows we will not always indicate the functional character of \underline{n} explicitly, but we will always assume that it is to be interpreted functionally.

Once we have determined that \underline{n} is a function, the question naturally arises as to which function \underline{n} is. In other words, how is the value of \underline{n} determined by " \underline{B} " in the simple case and by " \underline{C} " in the relativized case? Although \underline{n} plays a crucial role in the meanings of a number of plurality quantifiers, however, semantic theory does not have to provide an answer to this question. Since the value of \underline{n} in the semantic representation of any given *many*-quantification is determined by a wff that appears in that quantification, there is a strong temptation to conclude that the value of \underline{n} must be included in the lexical entry of that wff, when that wff is an atomic predicate, and must be provided by semantic rules in other cases. Since the value of \underline{n} that is determined by the predicate "Linguist" in (46) is different from the value of \underline{n} that is determined by the predicate "Study-Coptic" in (47), we might be led to conclude that the different values of \underline{n} must be included in the lexical entries of the respective predicates, for example (assuming they are atomic). This conclusion assumes, however, that the value of \underline{n} is related logically to the meaning of a predicate to which it corresponds, when, in fact, it is related empirically to the reference of the predicate.

The meaning of the predicate "Linguist" in (46) is something like "one who studies linguistics" or "one who studies language" and the meaning of the predicate "Study-Coptic" in (47) is something like "one who studies Coptic". Taken together with the fact that Coptic is a language, these meanings do logically entail that $\underline{n}(\text{Linguist})$ is at least as large as $\underline{n}(\text{Study-Coptic})$, that is, they do tell us something about the relative sizes of a manifold of linguists and a manifold of Coptic scholars. Without some empirical information about the actual state of society at a given time, however, these meanings tell us nothing about the actual values of \underline{n} . As far as logic is concerned, society could have developed in such a way that Coptic was the dominant language, in which case Coptic scholars would be in great demand and, therefore, in great supply as well. The meaning of the predicate "Study-Coptic" would not be different under such circumstances from what it is now. It would still mean something like "one who studies Coptic". The value of \underline{n} , however, would still be substantially larger than it happens to be today.

Since the semantic part of the lexical entry of a predicate is supposed to consist of all of the information that makes up the meaning of the predicate (within the context of a given grammar and general semantic theory), it follows that the values of \underline{n} for specific predicates do not have to be included in the lexicon. Since the value of \underline{n} for a specific predicate is determined empirically from the way in which the reference

of the predicate fits into the actual state of the world, the place for an explicit account of \underline{n} is in a formal theory of knowledge and beliefs, rather than a formal theory of linguistics or semantic competence. Some interesting ideas on how the two kinds of theories might interact in determining our concept of the world are discussed, in effect, in Putnam (1966).

Given the new semantic analyses (57), (58), and (59)/(60), the rest of Altham's simple quantifiers must be interpreted in terms of them, rather than in terms of (28), (29), and (30)/(31). Relativizing these quantifiers is straightforward, in terms of (49), (50), and (51)/(52).

For the first version of a *few* we get

$$(61) \quad (\underline{Fv})(\underline{B(v)}, \underline{A(v)}) = (\exists v)(\underline{B(v)}, \underline{A(v)}) \ \& \ -(\underline{Mv})(\underline{B(v)}, \underline{A(v)})$$

corresponding to (32), or

$$(\underline{Fv})(\underline{B(v)}, \underline{A(v)}) = (\exists v)(\underline{B(v)}, \underline{A(v)}) \ \& \ (\underline{Few v})(\underline{B(v)}, \underline{A(v)})$$

corresponding to (33). For *all but a few* we get

$$(\exists v)(\underline{B(v)}, \underline{A(v)}) = (\underline{Nv})(\underline{B(v)}, \underline{A(v)}) \ \& \ -(\forall v)(\underline{B(v)}, \underline{A(v)})$$

corresponding to (34).

For the second version of a *few* we get the explicit semantic analysis

An interpretation \underline{I} satisfies $(\underline{fv})(\underline{B(v)}, \underline{A(v)})$ iff there are at least $\underline{m(B)}$ distinct \underline{t} -variants of \underline{I} that satisfy $\underline{B(t)}$, all of which satisfy $\underline{A(t)}$,

corresponding to (35). Since \underline{m} , like \underline{n} , is really a function of the first wff in this analysis, it is also, like \underline{n} , a function of the wff in the simple analysis. It follows that (35) itself must be replaced by

An interpretation \underline{I} satisfies $(\underline{fv})\underline{A(v)}$ iff there are at least $\underline{m(A)}$ distinct \underline{t} -variants of \underline{I} , all of which satisfy $\underline{A(t)}$

as our analysis of simple *f*. *Exactly a few* can now be defined as either

$$(\underline{fv})(\underline{B(v)}, \underline{A(v)}) \ \& \ (\underline{Fv})(\underline{B(v)}, \underline{A(v)})$$

or

$$(\underline{fv})(\underline{B(v)}, \underline{A(v)}) \ \& \ -(\underline{Mv})(\underline{B(v)}, \underline{A(v)})$$

analogously to the simple case.

For *very nearly all* we get either the interdefinability statement

$$(\forall v)(\underline{B(v)}, \underline{A(v)}) = \sim(\underline{fv})(\underline{B(v)}, \sim \underline{A(v)})$$

corresponding to (36), or the explicit analysis

\underline{I} satisfies $(\forall v)(\underline{B(v)}, \underline{A(v)})$ iff in every $\underline{m(B)}$ distinct \underline{t} -variants of \underline{I} that satisfy $\underline{B(t)}$, at least one satisfies $\underline{A(t)}$

corresponding to (37). Again, (37) itself must be replaced by the functionalized rule

\underline{I} satisfies $(\forall v)\underline{A(v)}$ iff in every $\underline{m(B)}$ distinct \underline{t} -variants of \underline{I} , at least one satisfies $\underline{A(t)}$

Finally, we get the expression

$$(\forall v)(\underline{B(v)}, \underline{A(v)}) \ \& \ (\exists v)(\underline{B(v)}, \underline{A(v)})$$

corresponding to (38), as a definition of relativized *all but at most a very few*.

Section 4: Non-Reducibility of the Relativized Quantifiers to the Simple Case

The relativized versions of the plurality quantifiers *exactly n* and *at least n* can be reduced to their simple versions in the same way in which we reduced the relativized existential quantifier to its simple case in 11,1, 4. That sentences like

(62) *Exactly 2093 linguists have studied logic.*

At least 2093 linguists have studied logic.

are equivalent, respectively, to corresponding sentences like

(63) *Exactly 2093 people are linguists and have studied logic.*

At least 2093 people are linguists and have studied logic.

follows directly from our analysis of the simple existential quantifier in 11,1,2 and of the relativized existential quantifier in 11,1,3. The respective equivalence of the sentences in (62) to the sentences in (63) is analogous to the equivalence of formulas (25) and (26).

For the rest of our plurality quantifiers, however, such a reduction is impossible, regardless, it should be noted, of whether or not their analysis involves manifolds. In 11,1,4 we saw that a relativized existential quantification involving the wffs "B" and "A" can be made simple by replacing the ordered pair "(B,A)" with the single wff "BAA", but that the corresponding universal quantification can be made simple by replacing that ordered pair with "B \supset A". A careful examination of our analysis of relativized *many* in Section 3 reveals that *many* is analogous to *some* in this respect, rather than to *all*. If relativized *many* can be reduced to its simple case, then this can be done by replacing "(B,A)" with "BAA", rather than with "B \supset A".

The reason for this, in essence, is that *many*, like *some*, makes an existential statement, as we saw in connection with (53), (54), (55), and (56), rather than a conditional statement of the sort made by *all*. The sentence

(64) *Many linguists study logic.*

for example, is equivalent to the existential sentence

(65) *There are many linguists who study logic.*

just as the sentence

(66) *Some linguists study logic.*

is equivalent to the existential sentence

(67) *There are some linguists who study logic.*

In contrast, however, if we try to form a sentence with *all* that is parallel to (65) and (67), we get

**There are all linguists who study logic.*

which is not a sentence at all. In fact, there is no existential sentence to which the sentence

(68) *All linguists study logic.*

is equivalent.

Unlike (65) and (67), and thus (64) and (66), sentence (68) can be true even if there are no linguists at all and thus none who study logic, as long as any linguists who exist study logic. The *all* sentence, unlike the corresponding *many* and *some* sentences, makes no claim about the existence of logic-studying linguists. What it says is that whatever linguists there are, all of them study logic, even if there happen to be none. Sentence (68), in other words, is equivalent to the sentence

It is true of all people that if they are linguists then they study logic.

or, as we saw in 11,1,4, to the sentence

Everyone is such that if he is a linguist then he studies logic.

Both of these sentences are conditional, rather than existential, in character.

This is the reason that, as we saw in 11,1,4, relativized *all* is reduced by introducing " \supset ", rather than " \wedge ". Since *many*, like *some*, is existential, rather than conditional, like *all*, it follows that if its relativized case can be reduced to the simple case, this must be done by introducing a conjunction, rather than a conditional. A similar argument would show that if the quantifier *nearly all*, which is interdefinable with *many* in the same way in which *all* is interdefinable with *some*, is to have its relativized form reduced to its simple form, then this must be done, as with *all*, by introducing a conditional, rather than a conjunction.

As we noted earlier, however, such a reduction turns out to be impossible. We just saw that if the relativized quantification

(69) *Many linguists study logic.*

is reducible to a simple quantification, then the simple quantification it is reduced to will be

(70) *Many people both are linguists and study logic.*

In other words, if the relativized quantification

(71) (Many x)(B,A)

is to be reduced to a simple quantification, then the required simple quantification must be

(72) (Many x) ($B \wedge A$).

This result also makes sense intuitively, at least on the surface, as well as following from the foregoing argument.

When we examine (69) and (70) more carefully, however, we see that they are not equivalent at all. Suppose it is true that (71) is equivalent to (72). Since we can take either $B = \text{Linguist}$ and $A = \text{Study-logic}$ or $B = \text{Study-logic}$ and $A = \text{Linguist}$, it follows both that

(73) (Many x) ($\text{Linguist}(x), \text{Study-logic}(x)$)

is equivalent to

(74) (Many x) ($\text{Linguist}(x) \wedge \text{Study-logic}(x)$)

and that

(75) (Many x) ($\text{Study-logic}(x), \text{Linguist}(x)$)

is equivalent to

(76) (Many x) ($\text{Study-logic}(x) \wedge \text{Linguist}(x)$).

Since conjunction is commutative, the formulas

$\text{Linguist}(x) \wedge \text{Study-logic}(x)$

$\text{Study-logic}(x) \wedge \text{Linguist}(x)$

are equivalent, so (74) and (76) are equivalent. It follows that (73) and (75) are equivalent, since they are equivalent, respectively, to (74) and (76).

Formulas (73) and (75) are clearly not equivalent, however. Formula (73) is the semantic representation of

(77) *Many linguists study logic.*

and formula (75) is the semantic representation of

(78) *Many students of logic are linguists.*

Either of (77) or (78) could be true under conditions that would make the other false, depending on the values of $n(\text{Linguist})$ and $n(\text{Study-logic})$ and on the number of linguists and logic-students there are. In other words, many A's can be B's without many B's

being A's. Since our assumption that (71) and (72) are equivalent led to the conclusion that (73) and (75) are equivalent, when, in fact, they are not equivalent, it follows that the assumption is false. We cannot reduce the relativized quantification (71) to the simple quantification (72), because the two formulas are not logically equivalent.

Since relativized *many* cannot be reduced to its simple case, it follows that the rest of our manifold quantifiers, which can all be defined in terms of *many*, can also not be so reduced. The question remains, however, whether Rescher's and Kaplan's quantifiers *most* and *more than m/n* , whose definitions do not involve manifolds, can have their relativized versions reduced to the simple case. The reason that (77) and (78) are not equivalent is related to the fact that $n(\text{Linguist})$ is different, in general, from $n(\text{Study-logic})$. For there to be many linguists there must be *more than* $n(\text{Linguist}) - 1$ of them, but for there to be many logic-students there must be *more than* $n(\text{Study-logic}) - 1$ of them. *Most*, however, as we saw in 11.2.2, is equivalent to *more than $1/2$* for *any* wff or wffs. Its "index" does not vary, as does n for *many*, but is always simply $1/2$. Similarly, *each* quantifier *more than m/n* has the constant "index" m/n for all wffs, unlike *many*, which is systematically ambiguous with respect to its manifold size index n . We have seen that the systematically ambiguous plurality quantifiers cannot be reduced, but we can still ask whether *most* and *more than m/n* , which lack systematic ambiguity, can be reduced.

It is not difficult to see that this question must be answered in the negative. First we notice that both quantifiers, like *many*, make existential claims (even if $m = 0$, by the way). The sentence

(79) *Most linguists study logic.*

is equivalent to

(80) *More linguists study logic than do not.*

according to our analysis in 11.2.3.1, and (80), in turn, is equivalent to the sentence

(81) *More than fifty percent of linguists study logic.*

Neither (80) nor (81) can be true, if there are no linguists, and the same holds for analogous sentences with m/n -values other than fifty percent. It follows that, as we saw in the case of *many*, if relativized *most* or *more than m/n* are to be reduced to their simple forms, then it must be by introducing conjunction, rather than a conditional.

In this case, however, it is immediately obvious that such a reduction does not work. The sentence (79) makes a statement (81) about fifty percent of the class of linguists, but the purported conjunctive simple reduction

- (82) *More than fifty percent of all people are both linguists and students of logic.*

that is,

- (83) *Most people are both linguists and students of logic.*

makes a statement about fifty percent of the class of people. Sentences (82) and (83) entail that more than half of all people are linguists, but sentence (79) entails nothing about the size of the class of linguists except that it is non-empty. Since there are approximately three billion people in the world, (82) and (83) can be true only if more than one-and-one-half billion of them are linguists, but (79) will be true even if there are only a thousand linguists in the world, as long as more than five hundred of them study logic. It follows that *most* cannot be reduced by introducing a conjunction, with similar arguments and conclusions for other values of m/n .

One might still object that a conditional reduction of relativized *most* seems intuitively plausible, despite the fact that *most* makes an existential claim. Can we say that (79) is equivalent to the conditional sentence

- (84) *It is true of most people that if they are linguists then they study logic.*

as this objection suggests? It is not difficult to see that the respective semantic representations of these two sentences are not equivalent.

The semantic representation of (79) is of the form

- (85) $(\text{Most } y)(\text{Linguist}(y), \text{Study-logic}(y))$

and the semantic representation of (84) is of the form

- (86) $(\text{Most } y)(\text{Linguist}(y) \supset \text{Study-logic}(y)).$

The variable "y" is used in these formulas only because "x" is already used in the semantic analysis of *most* in (39) and (42), for a different purpose. Formula (86) is equivalent to the formula

- (87) $(\text{Most } y)(\neg \text{Linguist}(y) \vee \text{Study-logic}(y)),$

because the entailment

- (88) $\text{Linguist}(y) \supset \text{Study-logic}(y)$

is equivalent to the disjunction

- (89) $\neg \text{Linguist}(y) \vee \text{Study-logic}(y).$

Formula (86) is true if and only if formula (87) is true and the latter is the case if and only if (89) is true for most values of "x", that is, more than fifty percent. Formula (89) clearly is true for most values of "x", however, because most people happen not to be linguists. It follows that (88) is also true for most values of "x" and that both (87) and (86) are true.

Formula (85) itself, however, is clearly false, despite the truth of its purported reduction (86), because it says that most linguists study logic, which, unfortunately, is not the case. It follows that (85) can be false, when (86) is true, so the two formulas are not equivalent.

Someone might still persist in trying to find some way of reducing relativized *most* to its simple form, but no matter which of the other fourteen binary truth-functional connectives we try, we can always find a model that gives (85) and the proposed reduction different truth-values. Relativized *most* cannot be reduced to its simple form.

CHAPTER 3: PRESUPPOSITIONAL QUANTIFIERS

Section 1: The Notion of Presupposition

Keenan's (1971a) analysis of the "genuinely new and exciting quantifier" *only* depends crucially on the notion of presupposition, which was first introduced in the logical literature by Strawson (1952). As Strawson describes the relation of presupposition,

It is self-contradictory to conjoin S with the denial of S' if S' is a necessary condition of the truth, simply, of S . It is a different kind of logical absurdity to conjoin S with the denial of S' if S' is a necessary condition of the truth or falsity of S . The relation between S and S' in the first case is that S entails S' . We need a different name for the second case; let us say...that S presupposes S' .

(p. 18)

Keenan (1971b) elaborates on this notion as follows:

...a sentence S is said to be a logical consequence of a set of sentences S^* just in case S is true in every world (that is, under all the conditions) in which all the sentences of S^* are true. In such a case we also say that S follows logically from S^* , and that S^* logically implies S .

A sentence S logically presupposes a sentence S' just in case S logically implies S' and the negation of S , $\sim S$, also logically implies S' . In other words, the truth of S' is a necessary condition on the truth or falsity of S . Thus if S' is not true then S can be neither true nor false.

(pp. 45-46)

Horn (1969) was really the first to propose Keenan's analysis of *only*, but he failed to point out that it qualifies as a quantifier. He offers the following formalization of the idea of presupposition:

- a. If $(S \rightarrow S')$ and $(\sim S \rightarrow S')$, then S presupposes S' .
- b. If $(S \rightarrow S')$ and $(\sim S' \rightarrow \sim S)$, then S entails S' .

Rule b., of course, contains some redundancy, since $(S \rightarrow S')$

itself entails $(\sim S' \rightarrow \sim S)$, but its form parallels that of a. and so helps to clarify the difference between the two notions being defined.

In short, a sentence S entails a sentence S' , if we are justified in concluding S' , given S . S presupposes S' , if the very fact that S makes sense, that is, can be true or false, entitles us to conclude S' . In classical two-valued logic a. is equivalent to the statement that S' is logically true, because of the law of the excluded middle. In such a logic the only presuppositions are the logically true (valid) propositions and these are presuppositions of every proposition. It follows that the notion of presupposition can have non-trivial instances, only if we allow at least a third truth-value in our logic.

Van Fraassen (1971) develops a very sophisticated framework for the description of presuppositional languages and other languages that involve more than two truth-values. Our purposes require substantially less elegance and our discussion will, it is hoped, be linguistically more intuitive and revealing. Some authors, such as Wilson (1975), have recently objected to the notion of presupposition and suggested that entailment suffices to account for the facts of language. I will not comment on these suggestions, except to point out that the results of this chapter ipsis factis constitute empirical support for the linguistic reality of the presuppositional analysis.

One thing to notice about presupposition is its close similarity to relativization, which we have already examined in some detail. In van Fraassen's general framework, as we saw in Chapter 1, the truth-value of a quantificational sentence under an assignment function d is determined by examining the assignment functions d' that differ from d at most in their assignments to the variable of quantification. As we saw in Sections 3 of the last two chapters, the relativized versions of quantifiers are obtained by restricting the functions d' to those which satisfy some wff other than the one being quantified.

In the case of a sentence with a presupposition, somewhat similarly, the assignment functions d under which the sentence can be either true or false are restricted to those which satisfy some sentence other than the one whose truth-value is being determined. In both cases some assignment function d or set of assignment functions d' is restricted to the set of such functions that satisfy some wff or sentence other than the the main one involved.

Section 2: Keenan's Quantifiers

2.1: Keenan's Analyses

Keenan (1971a) gives an explicit semantic analysis for *only* and points out that the quantifiers *all but* and *someone plus* are interdefinable with it. Denoting *all but* by the symbolism " $(\forall x \text{ but } n')$ " and *someone plus* by the symbolism " $(\exists x \text{ plus } n')$ ", he gives us the following definitions:

$$(90) (\forall x \text{ but } n') b_x^n \text{ iff (only } x, n') -b_x^n$$

$$(91) (\exists x \text{ plus } n') b_x^n \text{ iff } \neg(\text{only } x, n') b_x^n$$

The symbols " n " and " n' " in (90) and (91) represent proper names and " b_x^n " is the result of replacing each occurrence of " n " in the closed formula " b " with the variable " x ".

This particular notation arises from Keenan's reluctance to recognize open formulas as legitimate semantic entities. Keenan calls the customary use of variables in open formulas

deplorable, for free occurrences of variables function semantically in a quite different way from bound occurrences. Namely, they function as names. One of the purposes of logical syntax is to assign symbols which are meaningful in different ways...to different grammatical categories, and to assign the same grammatical category to symbols which are meaningful in the same way. Thus, if a language is logical, we know how a symbol is interpreted and what kind of role it plays in the definition of truth once we know its grammatical category. But not to assign 'free' occurrences of variables a different grammatical category from bound occurrences undermines this attempt. Thus we use names - call them arbitrary names if you like - where some other people would use variables (calling them 'free' as a meta-linguistic afterthought).

(pp. 261-262, n.5)

This argument defeats itself, however, because exactly the same objection can be made to Keenan's distinction between names and arbitrary names as he just finished making to the distinction between free and bound (occurrences of) variables. Names serve to name, but arbitrary "names" serve as place holders, just as free variables do.

The real effect of pretending that free variables are names is to obscure the distinction between open and closed formulas, thus depriving us of any way to represent properties or relations independently of the individuals that have those properties or stand in those relations. As we saw on page 17, a sentence (closed formula) represents a proposition, while an open formula represents a relation (or property), without specifying what names can be truthfully substituted for the variables of the formula. One might argue that a third kind of symbol, completely different from either names or bound variables, should be introduced for use in open formulas in place of Keenan's "arbitrary names" or the more traditional free variables. I would not take issue with such an argument. To paraphrase Keenan, however, "not to assign 'arbitrary' occurrences of names a different grammatical category from non-arbitrary occurrences undermines this attempt (to assign symbols which are meaningful in different ways...to different grammatical categories). Thus we use variables - call them free variables if you like - where Keenan would use names (calling them 'arbitrary' as a metalinguistic afterthought)."

The form of *only* that occurs in (90) and (91) Keenan analyzes as follows:

- (92) For a given set of conditions (state of the world) C , $(\text{only } x, n') b_x^n$ is true under C just in case (a) and (b) below both hold. It is false just in case (a) holds but the sentence mentioned in (b) is false. Otherwise it has the third value zero.

(a) b_n^n is true under C (where b_n^n is the result of replacing each occurrence of n in b by n')

(b) $(\forall x)(b_x^n \supset (x=n'))$ is true under C .

He replaces the logician's technical notions of interpretation, assignment function, and model with the intuitive notion "set of conditions (state of the world)" and introduces a third truth-value, which he calls "zero", to give substance to the presuppositional character of (a). Condition (a) must hold, if the *only*-formula being analyzed is to be either true or false, rather than zero-valued. If (a) is true, then the truth-value of the *only*-formula is the same as that of (b). Condition (a) is the presupposition of the formula and (b) is its assertion, the sentence it is equivalent to, given that the presupposition is fulfilled.

Rule (92) accounts for the meanings of sentences like

(93) *Only Fermat could prove his last theorem.*

in which *only* occurs with a proper name. It tells us that (93) is true, if Fermat could indeed prove his last theorem, but no one else could; false, if both Fermat and someone else could prove the theorem; and of the third truth-value, neither true nor false, if Fermat could not prove it.

Keenan also gives the following analysis of sentences like

(94) *Only logicians can prove Gödel's theorem.*

in which *only* occurs with a common noun:

(95) For a given set of conditions C , $(\text{only } x, d_x^n)$ b_x^n is true under C just in case both (a) and (b) below hold: It is false as long as (a) holds and the base sentence mentioned in (b) is false. Otherwise it is zero valued.

- (a) $(\exists x)(d_x^n \ \& \ b_x^n)$ is true under C
- (b) $(\forall x)(b_x^n \supset d_x^n)$ is true under C .

Rule (95) tells us that (94) is true, if there are logicians who can prove Gödel's theorem, but no one else can; false, if there are both logicians and non-logicians who can prove the theorem; and of the third truth-value, if every logician proves unequal to the task. Again, Keenan defines *all but* and *someone plus* in terms of (95), as

$$\begin{aligned} (\forall x \text{ but } d_x^n) b_x^n &\text{ iff } (\text{only } x, d_x^n) - b_x^n \\ (\exists x \text{ but } d_x^n) b_x^n &\text{ iff } -(\text{only } x, d_x^n) b_x^n \end{aligned}$$

for sentences in which these quantifiers occur with common nouns.

Even aside from the peculiarity we discussed earlier, Keenan's notation is misleading, because it writes the proper name or common noun that occurs with the quantifier as a part of the quantifier, rather than as the first wff in a relativized quantification. His

- (96) $(\text{only } x, n') b_x^n$
- (97) $(\text{only } x, d_x^n) b_x^n$

should properly be written in the form

(98) $(\text{only } x)(x=n', b_x^n)$

(99) $(\text{only } x)(d_x^n, b_x^n)$

respectively, in which the two wffs in each relativized quantification are explicitly exhibited. It is easy to see from (98) and (99) that the former is simply a special case of the latter. Formula (98) arises from the replacement of the general wff symbol " d_x^n " in (99) with the specific open formula " $x=n'$ ". Looking back at the analyses (92) and (95), we see that the same relationship holds between them. Replacing " d_x^n " in (95b) with " $x=n'$ " automatically gives us (92b). Replacing " d_x^n " in (95a) gives us

$$(\exists x)(x=n' \ \& \ b_x^n),$$

which is just a cumbersome way of writing (92a). This fundamental relationship between (92) and (95), from which it follows that (92) is superfluous, given (95), is clearly revealed in (98) and (99), but is hidden from view in Keenan's notation (96) and (97).

Keenan's analyses (92) and (95) are as concealing as his notation. *Only* and the quantifiers that are interdefinable with it can occur only relativized, as we saw in 1,1,4. Relativization is an intrinsic part of their meaning. Rules (92) and (95), however, which purportedly give us the meaning of *only*, are formulated in terms of the simple reduced forms of the classical quantifiers that are involved in that meaning, rather than explicitly in terms of the relativized versions of those quantifiers. To fully capture the necessarily relativized character of *only*, (92b), if we use it at all, considering its superfluity, should be expressed in the relativized form

$$(\forall x)(b_x^n, x=n') \text{ is true under } C$$

and the conditions (a) and (b) of (95) should be written as

(100) (a) $(\exists x)(d_x^n, b_x^n)$ is true under C

(b) $(\forall x)(b_x^n, d_x^n)$ is true under C

Rule (100) reveals a symmetry in (a) and (b) that is hidden in Keenan's formulation (95), especially in view of the commutativity of " $\&$ ". It also reveals that what is going on in *only* is relativization, rather than the simple conjunction and entailment that (95) suggests.

Despite the weakness we have discussed in notation and formulation, Keenan's analyses (92) and (95) do accomplish, more or less, what they are supposed to. In particular, the fundamental insight that *only* is presuppositional is clearly expressed. Keenan uses this insight to formulate analyses of three more forms of *only*, each of which involves both an assertion and a presupposition.

The first of these is the quantifier *only n*, that is, *only* plus a natural number, which occurs in sentences like

- (101) *Only 63 logicians can prove Cohen's independence theorem.*

Keenan's analysis of this *only* is stated in terms of the natural-number quantifier, which he denotes by " (mx) " for the natural number m . As an analysis of this quantifier he gives us the rule

- (102) $(mx) b_x^n$ has the same truth value under all conditions C as does
- $$(\exists x_1)(\exists x_2) \dots (\exists x_m) ((x_1 \neq x_2) \& (x_1 \neq x_3) \& \dots \& (x_1 \neq x_m) \& (x_2 \neq x_3) \& \dots \& (x_2 \neq x_m) \& \dots \& (x_{m-1} \neq x_m) \& b_{x_1}^n \& b_{x_2}^n \& \dots \& b_{x_m}^n).$$

In other words $(mx)b_x^n$ has the same truth value as the sentence which says that there are m distinct things such that b holds of each one. (p.281)

Keenan's analysis of m in (102) is really an analysis of *at least m* and is somewhat more intuitive than the recursive definition of *at least n* that we saw in 11.2.1.2. It is also considerably more cumbersome than Altham's formulation, but is logically equivalent to it.

Keenan uses the quantifier defined in (102) to formulate the following definition of *only m*.

- (103) $(\text{only-}mx, d_x^n) b_x^n$ is true under C just in case both (a) and (b) hold. It is false if (a) holds and the sentence mentioned in (b) is true. Otherwise it is third valued.
- (a) $(mx)(d_x^n \& b_x^n)$ is true under C
- (b) $((m+1)x)(d_x^n \& b_x^n)$ is false under C .

As we would expect, (103) tells us that (101) is true, if there are exactly sixty-three logicians who can prove Cohen's theorem; false, if there are at least sixty-four logicians who possess that skill; and third-valued, if there are at most sixty-two.

Next Keenan considers sentences like

- (104) *Only some logicians are intuitionists.*

which contain the quantifier *only some*. He gives us the following analysis of that quantifier:

- (105) $(\text{only-}\exists x, d_x^n) b_x^n$ is true under C just in case both (a) and (b) below hold. It is false if (a) holds and the base sentence mentioned in (b) is false. Otherwise it gets the third value.
- (a) $(\exists x)(d_x^n \& b_x^n)$ is true under C
- (b) $-(\forall x)(d_x^n \supset b_x^n)$ is true under C .

Sentence (104) is true, according to (105), if there are both logicians who are intuitionists and logicians who are not; false, if all logicians are intuitionists; and third-valued, if no logicians are intuitionists.

As we saw in connection with (95) and (100), Keenan's notation is misleading and leads to analyses, in (103) and (105), just as much as in (92) and (95), that are less revealing than they would be. Keenan treats *only m* and *only some* as the simple reduced forms of relativized quantifiers, just as he treats *only* in (92) and (95). Unlike *only*, however, which can occur only relativized, both *only m* and *only some* can occur either simple or relativized. A purported sentence like

- (106) **Only things can be proven.*

in which *only* occurs simple, is semantically anomalous, at best, but the analogous sentences

- (107) *Only 29 things can be proven.*

- (108) *Only some things can be proven.*

in which *only m* and *only some*, respectively, occur simple, are both perfectly acceptable. In contrast to (106), the sentence

- (109) *Only theorems can be proven.*

in which *only* occurs relativized, is, as we saw in 1.1.4, as acceptable as (107) and (108), as are their relativized analogs

(110) *Only 29 theorems can be proven.*

(111) *Only some theorems can be proven.*

Since, as we have seen, Keenan's notation fails to distinguish at all between simple and relativized quantifiers, his analyses (103) and (105) fail to distinguish the acceptable simple quantifications in (107) and (108) from the semantically anomalous one in (106).

We can correct this defect in Keenan's analyses by providing both simple and relativized analyses for *only m* and *only some*, while still omitting a simple analysis of *only*. This solution may seem *ad hoc* at the moment, but the non-occurrence of a simple analysis for *only* will follow naturally from our discussion in IV.2.2. A principled relationship between the simple and relativized versions of quantifiers that have both will also emerge during our development of Part IV.

As a simple analysis of *only m* we get the rule

(112) (Only m x) A is true under C , if both (a) and (b) hold; false, if (a) holds and (b) does not; and third-valued otherwise.

(a) $(\exists_m x) A$

(b) $-(\exists_{m+1} x) A$

where " $\exists_m x$ " is the at least m quantifier defined for n in 1.1.2. *Only m* in (112) is different from Altham's *exactly m*, defined for n in 1.1.2, because it presupposes that there are m A 's, whereas Altham's quantifier asserts this. This seems to me to capture exactly the intuitive difference between *exactly n* and *only n*.

To get an analysis of simple *only some*, we replace "(Only m x)" in (112) with "(Only some x)" and the conditions (a) and (b), respectively, with

(113) (a) (Some x) A

(b) $-(\forall x) A$

Rule (112) tells us that (107) is true, if there are twenty-nine things that can be proven, but not thirty such things; false, if there are at least thirty things that can be proven; and third-valued, if there are twenty-eight or fewer. Rule (113) says that (108) is true, if some things can be proven and some things cannot; false, if everything can be proven; and third-valued, if nothing can be proven.

The relativized versions of these quantifiers are just as straightforward as the simple ones. Relativized *only m* can be analyzed in terms of the relativized at least n quantifier " $\exists_n x$ " that we saw in 1.1.2.3.1 by replacing "(Only m x) A " in (112) with "(Only m x) (B, A) " and conditions (a) and (b), respectively, with

(114) (a) $(\exists_m x)(B, A)$

(b) $-(\exists_{m+1} x)(B, A)$

This tells us that (110) is true, if there are twenty-nine, but not thirty, theorems that can be proven; false, if there are thirty theorems that can be proven; and third-valued, if there are twenty-eight or fewer such theorems.

To get an analysis of relativized *only some* we replace "(Only m x) A " in (112) with "(Only some x) (B, A) " and conditions (a) and (b), respectively, with

(115) (a) (Some x) (B, A)

(b) $-(\forall x)(B, A)$

This says that (111) is true, if some theorems can be proven and some theorems cannot; false, if all theorems can be proven; and third-valued, if no theorems can be proven.

The last case that Keenan considers is that of conjunctive *only*, which can occur only relativized, in sentences like

(116) *Only logicians and linguists get off on quantifiers.*

His analysis, as we might expect by now, takes the following form:

(117) (only $s, b_{1x}^n, b_{2x}^n, \dots, b_{mx}^n$) b_x^n is true under a given set of conditions C just in case both (a) and (b) below hold. It is false if (a) holds and the sentence in (b) is false. Otherwise it has value zero.

(a) $(\exists x)(b_{ix}^n \& b_x^n)$ is true in C , for each i between 1 and m .

(b) $(\forall x)(b_x^n \supset (b_{1x}^n \vee b_{2x}^n \dots \vee b_{mx}^n))$ is true under C .

Rule (117) tells us that (116) is true, if there are both logicians and linguists, but no one else, who get off on quantifiers; false, if there are logicians, linguists, and other people who appreciate the beauty of quantifiers; and third-valued, if either no logicians or no linguists can be found who do so. As we saw for each of the other forms of *only*, it can be reformulated in

explicitly relativized form by replacing conditions (a) and (b) with

- (118) (a) $(\exists x)(b_{ix}^n, b_x^n)$ is true in C , for each i between 1 and m
 (b) $(\forall x)(b_x^n, b_{ix}^n \vee b_{2x}^n \vee \dots \vee b_{mx}^n)$ is true in C ,

respectively.

2.2: The Semantic Effect of *Only*

Writing our semantic representations in explicitly relativized form enables us to see something about *only* that is not as clearly evident in Keenan's notation. If we take a careful look at our analyses of *only* and compare them to the sentences they are analyses of, we realize that the presupposition in each case can be obtained by simply omitting *only* from the sentence. Rule (112), for example, tells us that the presupposition of (107),

- (107) *Only 29 things can be proven.*

is

29 things can be proven.

Rule (113) tells us that the presupposition of (108),

- (108) *Only some things can be proven.*

is

Some things can be proven.

Similarly, rules (114) and (115), respectively, tell us that the presuppositions of (110) and (111),

- (119) *Only 29 theorems can be proven.*

Only some theorems can be proven.

are

29 theorems can be proven.

Some theorems can be proven.

respectively.

The necessarily relativized forms of *only* also fit this pattern, only a little more subtly. The case of *only* plus a proper name, if we choose to consider it separately, is almost straightforward. Rule (100) says that the presupposition of (98),

$(\text{only } x)(x=n', b_x^n)$

is

$(\exists x)(x=n', b_x^n)$

which, as we noted earlier, is equivalent to

b_n^n .

This means that the presupposition of (93),

Only Fermat could prove his last theorem.

is

Fermat could prove his last theorem.

in accordance with the pattern.

The case of *only* plus a common noun is somewhat less straightforward. Rule (100) tells us that the presupposition of

$(\text{only } x)(d_x^n, b_x^n)$

is

$(\exists x)(d_x^n, b_x^n)$

This means that the presupposition of (94),

Only logicians can prove Gödel's theorem.

is

(120) *Some logicians can prove Gödel's theorem.*

To fit our pattern, however, the presupposition of (94) should be

(121) *Logicians can prove Gödel's theorem.*

because this is what we get when we simply omit *only* from the sentence.

In fact, our pattern is preserved, because sentences like (121) are ambiguous. Sentence (121) can be read either as

(122) *All logicians can prove Gödel's theorem.*

or, perhaps a little less commonly, as (120). If (121) is used as a generic statement about the class of logicians, in answer to a request like *Tell me something about logicians*, then it is equivalent to (122). Sentence (121) can also be used, however, to answer a question like *Is there anyone who can prove Gödel's theorem?* When used in this way, (121) means the same thing as

There are logicians who can prove Gödel's theorem.

and is thus synonymous with (120). We see that (121), on one of its readings, is the presupposition of (94), again in accordance with the pattern we saw in *only m* and *only some*.

A similar analysis works for conjunctive *only*. Rule (118) says that both of the sentences

(123) *Logicians get off on quantifiers.*

(124) *Linguists get off on quantifiers.*

on their readings that are analogous to (120), as a reading of (121), are presuppositions of (116).

(116) *Only logicians and linguists get off on quantifiers.*

It follows that the conjunction of (123) and (124), namely,

(125) *Logicians and linguists get off on quantifiers.*

is also a presupposition of (116).

We see now what all forms of *only* have in common, despite the different analyses that we (based on Keenan) have given for them. In each case the presupposition of a sentence that contains *only* can be obtained by simply removing the *only*. The semantic effect of *only* is to "reaffirm" the truth of the sentence to which it is added and then to assert that the extension of the wff quantified in that sentence is limited to the classes that appear in the sentence. *Only* in (116) reaffirms (125) and goes on to say that no one else gets off on quantifiers. *Only* in (110) reaffirms (119) and goes on to say that there are no further theorems that can be proven. Similar statements can be made about the other examples we have considered.

In particular, rule (100) tells us that the assertion of (99) is

(126) $(\forall x)(b_x^n, d_x^n)$

so the assertion of (94)

Only logicians can prove Gödel's theorem.

is

(127) *Anyone who can prove Gödel's theorem is a logician.*

Formula (126), however, is equivalent to

$(\text{No } x)(\neg d_x^n, b_x^n)$

because of interdefinability and the commutativity of conjunction. This equivalence tells us that

No one who is not a logician can prove Gödel's theorem.

is the assertion of (94). Again we see that the effect of *only* once it has reaffirmed, through presupposition, the sentence it is added to, is to assert that the extension of the second wff is limited to that of the first. This is the real significance of the symmetry that we noted in connection with (100).

Section 3: Cushing's Quantifiers

In the last section we saw that the semantic effect of *only* is to reaffirm as a presupposition the sentence to which it is added and to assert further that the extension of the second wff of that sentence is restricted to that of the first wff, as specified by the sentence itself. The quantifier *also* has a somewhat different effect. The sentence

(128) *Also linguists study quantifiers.*

for example, presupposes that there are people other than linguists who study quantifiers and asserts that at least some linguists do.

The effect of adding *also* to the sentence

(129) *Linguists study quantifiers.*

is to assert that (129) is true, on the non-generic reading we discussed in connection with (121), while informing us through presupposition that the second wff (in this case the predicate "Study-quantifiers") of (129) holds of someone other than linguists, as well. Whereas *only* presupposes the sentence to which it is added, on its non-generic reading, *also* asserts that sentence on that reading. Whereas *only* asserts that the extension of the

second wff of that sentence has no members other than those mentioned in the sentence, also presupposes that there are other such members. It does not inform us, however, as to who or what they are.

Cushing (1972a) argues that *also* can be analyzed in Keenan-style terms as follows:

(130) (Also x)(B,A) is true if both (a) and (b) hold; false if (a) holds but (b) does not; and third-valued otherwise.

(a) $(\exists x)(\neg B, A)$

(b) $(\exists x)(B, A)$

in accordance with the foregoing discussion. Something like "under conditions C" should be understood wherever relevant in (130) and forthcoming analyses.

Both intuitively and from (130) we can see that *also* is interdefinable with at least one other quantifier. If we deny the sentence (128), we get the sentence

(131) *It is not the case that also linguists study quantifiers.*

which says the same thing as the sentence

(132) *Only someone other than linguists studies quantifiers.*

This sentence presupposes the appropriate instance of (130a) and denies the corresponding instance of (130b), just as we would expect of the denial of (128).

Formally this gives us the interdefinability relationship

(133) only some...other than = not also

where "... " stands for *-one*, *-thing*, *-where*, or whatever other simple-quantification suffixes might exist. Cushing (1972a) writes the quantifier on the left side of (133) as *in contrast some*, which has the advantage of not requiring the indefinite "... " in its formulation. *Only some...other than* seems to capture the meaning involved more exactly, however, than does *in contrast some*. Combining (130) and (133) gives us

(134) (Only some...other than x)(B,A) is true if both (a) and (b) hold; false if (a) holds but (b) does not; and third-valued otherwise.

(a) $(\exists x)(\neg B, A)$

(b) $(\forall x)(B, \neg A)$

as an explicit semantic analysis of this quantifier. Again we see a certain symmetry in the wffs in the analysis, as we saw in (100) for *only*.

Keenan (1971a) argues that *also* is not a quantifier, because it violates what he calls the "predication property". His description of the property itself is very vague, but we can get some idea of what Keenan means by looking at his argument against *also*. Keenan claims that

also does not meet the predication test in that the denial of (58a) below does not deny merely the meaning of *also* but rather goes all the way and actually denies (58c) itself.

(58a) *John also left.*

(58b) *It is not the case that John also left.*

(58c) *John left.*

Had *also* operated strictly as a predication functor we might have expected (58b) to still imply (58c) but deny merely that John's leaving was in addition to someone else's. But this is not the case.

By this test, however, even *some* fails to qualify as a quantifier.

Keenan argues, in effect, that *also* is not a quantifier, because the sentence

It is not the case that John also left.

$\neg(\text{Also } x)(x=\text{John}, \text{Left}(x))$

implies (entails) the sentence

(135) *It is not the case that John left.*

$\neg\text{Left}(\text{John}).$

The sentence

It is not the case that some man left.

$\neg(\text{Some } x)(\text{Man}(x), \text{Left}(x))$

however, also entails (135). Even more to the point, it entails the formula

-Left(c)

for every man c, not just John. It follows that Keenan's reason for denying quantifier status to *also* applies even more strongly to *some*, which is unquestionably a quantifier; if anything is. If we wish to admit *some* as a quantifier, we must also accept *also*, pending more substantial arguments against it.

Where Keenan went wrong in this argument is in failing to realize that *also* with a proper name is just a special case of *also* with a common noun. There is nothing particularly strange or "unquantificational" about the fact that the denial of *also* involves the assertion (134b), but this is all that Keenan's objection to *also* amounts to. When we take

$B = (x = \text{John})$

the formula (134b) reduces to (135), which Keenan objects to, but (134b) is still all it really is. I suspect that what really bothers Keenan about *also* is the appearance of "B" in (130a) and I agree that it might seem strange that a necessarily relativized quantifier that is itself relativized to "B" should have a quantifier relativized to "B" as its presupposition. This is simply a peculiarity of *also*, however, and has nothing to do with the question of quantifiers.

The fact that *also* is a quantifier is underscored by the fact that its negation in English involves the Keenan-recognized quantifier *only*, as a part of *only some...other than*. Keenan's (58b) can be reformulated as the synonymous sentence

Only someone other than John left.

and the semantic analysis of *only some...other than* also involves relativization to the negation of the wff to which the quantifier itself is relativized, as seen in (134a).

Sentence (132), in fact, is equivalent to the sentence

(136) *Only non-linguists study quantifiers.*

which is representable semantically by the formula

(137) $(\text{Only } x)(\neg \text{Linguist}(x), \text{Study-quantifiers}(x)).$

According to our analysis of *only*, based on Keenan's, in Section 2.1, formula (137) is analyzable semantically as the presupposition and assertion, respectively

(138) (a) $(\exists x)(\neg \text{Linguist}(x), \text{Study-quantifiers}(x))$

(b) $(\forall x)(\text{Study-quantifiers}(x), \neg \text{Linguist}(x)),$

as an instance of (100). Because of the equivalence of contrapositives, moreover, we know that

$\text{Study-quantifiers}(x) \supset \neg \text{Linguist}(x)$

and

$\text{Linguist}(x) \supset \neg \text{Study-quantifiers}(x)$

are logically equivalent, so we can reformulate (138) as

(139) (a) $(\exists x)(\neg \text{Linguist}(x), \text{Study-quantifiers}(x))$

(b) $(\forall x)(\text{Linguist}(x), \neg \text{Study-quantifiers}(x))$

by noticing that the simple reduced forms of the two (b) conditions are equivalent. Formulas (139), however, are the analysis of (132) that results from (134). We see that Keenan's own analysis (in our notation) of *only* gives us the same results as our analysis of the negation of *also*. This constitutes very strong evidence, given the correctness of Keenan's treatment of *only*, for both the correctness of our analysis of *also* and the quantificational character of *also*. It is hardly likely that *also* is not a quantifier, when its negation or denial so clearly is.

Section 4: Definite Descriptions and Proper Names

As we saw in the last section, Keenan's basic approach can be used to construct semantic analyses for quantifiers other than those he considers himself, other, even, than those he would recognize as quantifiers at all. As a further example, we can construct an analysis of *only a few* by replacing "(Only m x) A" in (112) with "(Only a few x) A" and conditions (a) and (b), respectively, with

(140) (a) $(A \text{ few } x)(B, A)$

(b) $\neg (\text{Many } x)(B, A)$

where "(A few x)A" is defined along the lines of 11,2,3. Again we see that the presupposition of a sentence like

Only a few linguists understand quantifiers.

is obtained by simply removing the *only*, to get, in this case

A few linguists understand quantifiers.

Only a few differs from Altham's *exactly a few*, whose simple version we saw in 11,2.1.2 and whose relativized version we saw in 11,2.3.2, in exactly the same way as Keenan's *only m* differs

from the n instance of Altham's *exactly n*. Whereas *exactly a few* asserts *a few*, *only a few* presupposes it. Both quantifiers assert *not many*. An analysis of a simple *only a few* is easily obtained by replacing "(B,A)" in (140) with "A" and taking it as an analysis of "(Only a few x)A".

As we might expect from their treatment in Zulu, which we examined in 1.1.3, the "quantitative pronouns" *all n* can also be given semantic analyses very similar to that of *only*. A sentence that contains *all n* presupposes that there are exactly n individuals that satisfy its first wff and asserts that all of those individuals satisfy its second wff. A sentence like

(141) *All six theorems have been proven.*

presupposes that there are exactly six theorems (that are of interest in the given context) and asserts that every one of them has been proven. A sentence like

(142) *Both theorems have been proven.*

which contains the suppletive form *both* of the non-occurring **all two*, presupposes that there are exactly two (relevant) theorems and asserts that each of them has been proven.

It follows that we can analyze *all n* semantically in terms of the following rule:

(143) (All n x)(B,A) is true if (a) and (b) both hold; false if (a) holds and (b) does not; and third-valued otherwise.

(a) $(\exists x)_n B$

(b) $(\forall x)(B,A)$

Rule (143) tells us that (141) is true, if there are exactly six theorems and all of them have been proven; false, if there are exactly six theorems and at least one of them has not been proven; and third-valued, neither true nor false, if there are fewer than six theorems or more than six theorems. It also tells us that (142) is true, if there are exactly two theorems and each of them has been proven; false, if there are exactly two theorems and at least one of them has not been proven; and third-valued, if there are fewer or more than two theorems.

Rule (143) seems to be the correct analysis of *all n* for every value of n greater than or equal to 2. These are the values of n for which the relevant "quantitative pronouns" exist in Zulu and for which *all n* exists in English, if we

recognize *both* as the occurring form of **all two*. The most natural thing to do now is to try to generalize, that is, to see what happens to rule (143) when we try to use it for other values of n .

Taking $n=0$ in (143) is not very interesting, because it presupposes that there are no B's and then tells us that whatever B's there are have A. Taking $n=1$, however, turns out to be quite interesting. Letting $n=1$ leaves (b) unchanged, but it turns (a) into

(144) $(\exists x) B.$

If we expand (144) explicitly in the form in which we analyzed it in 1.1.2, but using our simplified notation, rather than Altham's, we see that (144) is just an abbreviation for the formula (a) of

(145) (a) $(\exists x)(B(x) \wedge \neg(\exists y)(B(y) \wedge y \neq x))$

(b) $(\forall x)(B,A)$

Formula (b) has just been carried over from (143), so (145) is the case of (143) in which $n=1$. This makes (145a) and (145b) the presupposition and assertion, respectively, of the formula "(All one x)(B,A)".

The question now is "Is (145) the semantic analysis of any real sentence (in English, for example), and, if so, which ones?" We can answer this question by looking carefully again at (141) and (142). These sentences are synonymous, respectively, with the sentences

(146) *The six theorems have been proven.*

(147) *The two theorems have been proven.*

Since (146) and (147) are synonymous, respectively, with (141) and (142), they must have the same semantic analyses (143), with $n=6$ and $n=2$, respectively. It follows that taking $n=1$ in (143), if it yields any English sentence at all, should amount to the same thing as replacing the "six" in (146) and the "two" in (147) with "one". This gives us

(148) *The one theorem has been proven.*

as the sentence of which (145) is the semantic analysis.

We are not finished, however. Sentence (148), in fact, is just an awkward, perhaps emphatic, way of saying the more natural sentence

(149) *The theorem has been proven.*

Since (149) means exactly the same thing as (148), emphasis and stylistics aside, it must have the same semantic representation and analysis as (148). It follows that (145) is the semantic analysis of (149), as well as of (148).

It follows, in other words, that (145) gives us the semantic analysis of the definite article *the*, when it occurs in the singular. We can formalize this fact by rewriting (145) explicitly as an analysis of *the*, as follows:

(150) (The x)(B,A) is true, if (a) and (b) both hold; false, if (a) holds but (b) does not; and third-valued otherwise.

(a) $(\exists x)(B(x) \wedge \neg(\exists y)(B(y) \wedge y \neq x))$

(b) $(\forall x)(B,A)$

Rule (150) says that (149) is true, if there is exactly one theorem (in the given context) and that theorem has been proven; false, if there is exactly one theorem and that theorem has not been proven; and third-valued, neither true nor false, if there is no theorem or more than one.

An analysis of plural *the* can be obtained very easily from (150) by replacing (a), which says *exactly one*, with the corresponding formula for *at least two*. This gives us the rule

(151) (The x's)(B,A) is true, if (a) and (b) both hold; false, if (a) holds, but (b) does not; and third-valued, otherwise.

(a) $(\exists_2 x) B$

(b) $(\forall x)(B,A)$

where (a) is to be interpreted as in 11,2,1.2. According to (151), the sentence

The theorems have been proven.

is true, if there are at least two theorems and each of them has been proven; false, if there are at least two theorems and at least one of them has not been proven; and third-valued, if there is only one theorem or if there are no theorems.

Rule (150) gives us the semantic analysis of singular *the*, but it also gives us a lot more than that. There is nothing in (150) that limits B to the class of common nouns. "B" in (150) denotes, in fact, any wff at all. If we replace "B" in (150) with the formula

(152) $\text{Theorem}(x) \wedge \text{Fermat-discovered}(x)$

for example, we get the formula

(153) $(\text{The } x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x), \text{Proven}(x)).$

According to rule (150), the presupposition and assertion, respectively, of (153) are

(154) (a) $(\exists x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x) \wedge \neg(\exists y)(\text{Theorem}(y) \wedge \text{Fermat-discovered}(y) \wedge y \neq x))$

(b) $(\forall x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x), \text{Proven}(x)).$

Formula (153), however, is the semantic representation of the sentence

(155) *The theorem that Fermat discovered has been proven.*

if we interpret it analogously to how we have interpreted all of our other relativized quantification formulas. It follows that (154) gives us the presupposition and assertion, respectively, of (155), as well as of (153). This tells us that (155) is true, if Fermat discovered exactly one theorem and that theorem has been proven; false, if Fermat discovered exactly one theorem and that theorem has not been proven; and third-valued, neither true nor false, if Fermat discovered other than one theorem, that is, either at least two or none at all. This seems to me to be exactly the intuitive meaning of (155).

Expressions like

(156) *the theorem that Fermat discovered*

in (155) are what logicians have traditionally called "definite descriptions". The expression (156) itself would normally be written in terms of the definite-description operator "*i*", as

(157) $(ix)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x))$

and the sentence (155) would be written as

(158) $\text{Proven}((ix)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x)))$

with (157) as the argument of "Proven". The symbol "*(ix)*" can be interpreted as the unique x such that and is the traditional

logical representation of our "(The x)" in (150), the formal semantic form of the singular *the*. In view of the fact that (156) can be analyzed semantically in terms of (150) and represented semantically as (153), we see that (158) can be reformulated explicitly as a relativized quantification involving the two wffs (152) and "Proven(x)". This gives us

- (159) $(\exists x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x), \text{Proven}(x))$

as an equivalent reformulation of (158). We see that the traditional definite-description operator is really a necessarily relativized presuppositional quantifier, just like *only*.

Definite descriptions have been discussed extensively in the logical literature, since they were first discussed explicitly by Russell (1919). Russell was the first to recognize that definite descriptions are incomplete symbols, that is, that they can be defined only in terms of the sentences in which they occur. He proposed, in essence, that a sentence like (155) should be analyzed as a conjunction, in this case, of (154a) and (154b). Strawson (1950) disputes Russell's analysis and proposes, instead, that (155) should be analyzed as presupposing something like (154a) and asserting some version of (154b). The results of our discussion of (150) can now be seen as independent evidence in support of Strawson's analysis and against Russell's.

Let us review our argument briefly. We began by examining Keenan's analyses of various forms of *only* and found that, despite weaknesses in notation and formulation, they do account correctly for the intuitive meanings of those expressions. We then noted that *only* is treated in Zulu as one of a class of "quantitative pronouns", which also happens to include expressions of the form *all n* for values of *n* greater than or equal to 2. Since *all n* is thus capable of being treated morphologically in a natural language as the same kind of thing as *only*, we then suggested that it might also be possible to treat it semantically in that way. Since we had found no serious fault with the content of Keenan's semantic analyses of *only*, we constructed an analysis of *all n* within his framework and found that this analysis did indeed correctly express the meaning of that expression. This gave us analyses of *all n* for all values of *n* equal to or greater than 2, which are the values for which the "quantitative pronouns" exist in Zulu and for which the forms *all n* occur in English (with *both* as a suppletive form of **all two*). To maximize generality we asked what sense, if any, it would make to take *n=1* in our analysis of *all n* and we found, upon doing this, that an analysis of definite noun phrases results. The definite article turned out, in effect, to be a suppletive form of **all one*, just as *both* is a suppletive form of **all two*. Reexamining

our analysis, we realized that it is an analysis not only of definite noun phrases, but of definite descriptions in general, and that it coincides with Strawson's proposed presuppositional analysis of definite descriptions, in contrast to Russell's conjunctive proposal. Sentences involving definite descriptions turned out to be necessarily relativized quantifications with the presuppositional definite-description operator as the quantifier.

It is worth noting here that sentences that involve proper names can also be analyzed as quantifications, because they can be analyzed in terms of definite descriptions. The sentence

- (160) *Edward Keenan is a linguist.*

for example, is equivalent to the sentence

The unique x such that x=Edward Keenan is a linguist.

and so can be represented semantically as either

- $(\exists x)(x=\text{Edward Keenan}, \text{Linguist}(x))$

or

- (161) $(\text{The } x)(x=\text{Edward Keenan}, \text{Linguist}(x))$

This is exactly the same phenomenon that we saw in connection with (96), (97), (98), and (99), where we saw that *only* with a proper name is really just a special case of *only* with a common noun. Rule (150) says that (161) is true, if there is exactly one person who is Edward Keenan and that person is a linguist; false, if there is exactly one person who is Edward Keenan but that person is not a linguist; and third-valued, neither true nor false, if there is either no one or more than one person who is Edward Keenan. This seems to me to be a correct account of the meaning of (160).

Section 1: Generalizations of Relativization

1.1: Definite Descriptions and the Classical Quantifiers

In the last section we saw that sentences that involve definite noun phrases and sentences that involve definite descriptions in general are both analyzable as necessarily relativized presuppositional quantifications. When we compare the two kinds of sentences, however, an important difference emerges. A sentence like

(162) *The theorem has been proven.*

can be represented semantically as

(163) $(\text{The } x)(\text{Theorem}(x), \text{Proven}(x))$

and the sentence

(164) *The theorem that Fermat discovered has been proven.*

can be represented semantically as

(165) $(\text{The } x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x), \text{Proven}(x))$

Formula (163) has the presupposition and assertion, respectively,

(166) (a) $(\exists x)(\text{Theorem}(x) \wedge \neg(\exists y)(\text{Theorem}(y) \wedge y \neq x))$

(b) $(\forall x)(\text{Theorem}(x), \text{Proven}(x))$

and formula (165) has the presupposition and assertion, respectively,

(167) (a) $(\exists x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x) \wedge \neg(\exists y)(\text{Theorem}(y) \wedge \text{Fermat-discovered}(y) \wedge y \neq x))$

(b) $(\forall x)(\text{Theorem}(x) \wedge \text{Fermat-discovered}(x), \text{Proven}(x))$

Both (166) and (167) follow from the analysis of the in (150).

The only difference between (162) and (164) is the fact that (164) contains the restrictive relative clause *that Fermat discovered* and (162) does not. As we saw in 1.1.4, however, a quantifier with a restrictive relative clause is equivalent to a relativized quantifier. In this case we can interpret (164) either as the relativization of (162) to the wff "Fermat-discovered" or the relativization of the sentence

The thing Fermat discovered has been proven.

to the wff "Theorem". In the first case the set of theorems is

restricted to those that Fermat discovered and in the second case the set of things discovered by Fermat is restricted to those that are theorems. The two alternatives are logically equivalent and differ only in form. We will see that a very natural generalization of the definite-description can be developed, if we adopt the latter alternative.

Formula (165) can be reformulated in a way that explicitly reveals the relativization of "Fermat-discovered" to "Theorem". This gives us

(168) $(\text{The } x)(\text{Theorem}(x), \text{Fermat-discovered}(x), \text{Proven}(x))$

as a more revealing semantic representation than (165) and

(169) (a) $(\exists x)(\text{Theorem}(x), \text{Fermat-discovered}(x) \wedge \neg(\exists y)(\text{Theorem}(y), \text{Fermat-discovered}(y) \wedge y \neq x))$

(b) $(\forall x)(\text{Theorem}(x), \text{Fermat-discovered}(x), \text{Proven}(x))$

as the presupposition and assertion, respectively, of (169). Formula (169a) presents no problem of interpretation, because it involves only the kind of relativization we have already discussed. It is constructed by relativizing both the formula

$(\exists y)(\text{Fermat-discovered}(y) \wedge y \neq x)$

and the formula

$(\exists x)(\text{Fermat-discovered}(x) \wedge \neg(\exists y)(\text{Theorem}(y), \text{Fermat-discovered}(y) \wedge y \neq x))$

to the wff "Theorem".

Formula (169b), however, does pose a problem, because we have not yet developed a way to interpret universal quantifications that involve three wffs. We can figure out how to interpret such formulas by remembering that (169b) is supposed to mean the same thing as (167b) and that both express the assertion of (164). Sentence (164) presupposes that there is one and only one theorem that Fermat discovered, as expressed in (167a) and (169a), and asserts that whatever theorem Fermat discovered, that theorem has been proven. The most natural way to write this assertion formally is as the formula

(170) $(\forall x)((\text{Fermat-discovered}(x) \wedge \text{Theorem}(x)) \supset \text{Proven}(x))$

that is, the assertion that if x is both a thing discovered by

Fermat and a theorem, then it has been proven (for all values of "x"). Formula (170) clearly expresses the intended meaning of (167b) and (169b).

What is even more important for our purposes, however, is that (170) expresses this meaning in a way that is very similar to the way in which we reduced relativized *all* to simple *all* in 11,1,4. We saw there that a relativized universal quantification can be turned into an equivalent simple one by replacing the ordered pair of wffs "(B,A)" with the single wff " $B \supset A$ ", which represents a conditional or entailment. In such a case "A" is the main wff of the quantification and "B" is the wff to which the quantification is relativized. In the three-way quantification (168), similarly, there are two wffs to which the quantification of the third wff is relativized. Again we can reduce this relativization by introducing an entailment with both relativization wffs as antecedents and the main wff as consequent. Instead of replacing "(B,A)" with " $B \supset A$ ", in other words, we replace "(C,B,A)" with " $(C \wedge B) \supset A$ ". We see that definite descriptions are not, strictly speaking, relativized quantifiers, as we said they were in 11,3,4, but are more accurately described as a generalized form of quantification that involves three wffs, one primary and two to which that one is relativized.

The meaning of such a generalized quantifier can be determined either through reduction, as we just saw, or explicitly in terms of a double restriction on the Kaplan or van Fraassen assignment functions or the Mendelson Σ sequences that we use to determine the truth-values of quantificational sentences. In fact, we can completely generalize the universal quantifier by permitting any finite number of main wffs and any finite number of relativization wffs, distinguishing the two kinds of wffs in semantic representation by separating them with a semi-colon, rather than a comma. In van Fraassen's framework this gives us a semantic analysis like the following:

- (171) $\underline{M} \models (\text{All } x)(B_1, \dots, B_n; A_1, \dots, A_m)[\underline{d}]$ iff
 $\underline{M} \models (A_j[\underline{d}']), j=1, \dots, m$, for all assignments \underline{d}' for \underline{M} which are like \underline{d} except perhaps at x and for which $\underline{M} \models B_i[\underline{d}']$, $i=1, \dots, n$.

Rule (171) provides us with semantic analyses of sentences like

- (172) *Every theorem that was discovered by Fermat that we know about has been proven and published.*

for which $n=3$ and $m=2$. The semantic representation of (172) is the formula

- (173) $(\text{All } x)(\text{Known}(x), \text{Fermat-discovered}(x), \text{Theorem}(x); \text{Proven}(x), \text{Published}(x))$

the truth conditions of which are given by (171). The truth conditions can also be determined from the simple reduced quantification

- $(\text{All } x)((\text{Known}(x) \wedge \text{Fermat-discovered}(x) \wedge \text{Theorem}(x)) \supset (\text{Proven}(x) \wedge \text{Published}(x)))$

because of the equivalence

- (174) $(\text{All } x)(B_1, \dots, B_n; A_1, \dots, A_m)$ has the same truth-value (under f, \underline{d}, C , etc.) as
 $(\text{All } x)((B_1 \wedge \dots \wedge B_n) \supset (A_1 \wedge \dots \wedge A_m))$

which follows directly from (171). Either (171) or (174) can be considered as giving the meaning of generalized *all*.

A generalized existential quantifier can be defined analogously to the universal one defined in (171) to give us semantic analyses of sentences like

- (175) *Some theorems that Fermat discovered that we know about have been proven and published.*

The meanings of such sentences can be accounted for by the van Fraassen-style analysis

- (176) $\underline{M} \models (\text{Some } x)(B_1, \dots, B_n; A_1, \dots, A_m)[\underline{d}]$ iff
 $\underline{M} \models A_j[\underline{d}']$, $j=1, \dots, m$, for at least one assignment \underline{d}' for \underline{M} which is like \underline{d} except perhaps at x and for which $\underline{M} \models B_i[\underline{d}']$, $i=1, \dots, n$

or by the equivalence

- (177) $(\text{Some } x)(B_1, \dots, B_n; A_1, \dots, A_m)$ is true if and only if $(\text{Some } x)(B_1 \wedge \dots \wedge B_n \wedge A_1 \wedge \dots \wedge A_m)$ is true

which follows directly from (176). An analysis and reduction equivalence for *no* can be obtained by replacing "Some" in (176) and (177) with "No" and "at least one" in (176) with "no".

Now that we have analyses for the generalized universal and existential quantifiers, we can easily construct an analysis of

the definite description as a generalized quantifier as well. The generalized definite description appears in sentences like

- (178) *The theorem that Fermat discovered that we know about has been proven and published.*

and requires three kinds of wffs in its semantic analysis and representation, in order to account properly for its presuppositions and assertions.

First we notice that (169a) is most properly formulated as a generalized quantification, now that (176) is available to us. We can reformulate (169a) as the formula

- (179) (a) $(\exists x)(\text{Theorem}(x); \text{Fermat-discovered}(x),$
 $-(\exists y)(\text{Theorem}(y); \text{Fermat-discovered}(y),$
 $y \neq x))$

whose meaning can readily be obtained from (176), if we distinguish carefully between the commas and the semi-colons. Formula (169b), similarly, should be reformulated as the formula

- (179) (b) $(\forall x)(\text{Theorem}(x), \text{Fermat-discovered}(x);$
 $\text{Proven}(x))$

in which the two different kinds of wffs are again distinguished by a semi-colon.

Both (179a) and (179b) can be analyzed in terms of (176) and (174), respectively, because each involves only two kinds of wffs, main wffs and relativization wffs. In (179a) the wffs "Fermat-discovered(y)" and " $y \neq x$ " are both relativized to the wff "Theorem(y)" and the wffs "Fermat-discovered(x)" and " $-(\exists y)(\text{Theorem}(y); \text{Fermat-discovered}(y), y \neq x))$ " are both relativized "Theorem(x)". In (179b) the wff "Proven(x)" is relativized to both of the wffs "Theorem(x)" and "Fermat-discovered(x)". In neither case is there anything that is not accounted for by our analyses (176) and (174).

When we combine (179a) and (179b) as the presupposition and assertion, respectively, of (164) and (168), however, we see that the wff "Fermat-discovered(x)" plays a dual role. In (179a) "Fermat-discovered(x)" is a main wff, relativized to "Theorem(x)", but in (179b) it is a relativization wff, one of the wffs to which "Proven(x)" is relativized. A general analysis of sentences like (164) must clearly distinguish such dual-role wffs from both the relativization wffs and the main wffs of those sentences.

It follows that we should semantically represent sentences like (164) and (178) by formulas of the form

- (180) $(\text{The } x)(C_1, \dots, C_{n_1}; B_1, \dots, B_{n_2}; A_1, \dots, A_{n_3})$

in which the C_k are the relativization wffs, the B_j are the dual-role wffs, and the A_i are the main wffs. Adopting our van Fraassen-style analyses for generalized *all* and *some*, we return to Keenan's framework and get the following analysis of generalized *the*:

- (181) $(\text{The } x)(C_1, \dots, C_{n_1}; B_1, \dots, B_{n_2}; A_1, \dots, A_{n_3})$ is
 true under a given set of conditions, if both (a)
 and (b) hold; false, if (a) holds but (b) does
 not; and third-valued otherwise.
- (a) $(\exists x)(C_1(x), \dots, C_{n_1}(x); B_1(x), \dots, B_{n_2}(x),$
 $-(\exists y)(C_1(y), \dots, C_{n_1}(y); B_1(y), \dots,$
 $B_{n_2}(y), y \neq x))$
- (b) $(\forall x)(C_1, \dots, C_{n_1}, B_1, \dots, B_{n_2}; A_1, \dots, A_{n_3}).$

Again, the key to understanding (181) is the placement of the semi-colons. The presupposition, (a), relativized the B_j and a supplementary wff, first the one that begins with " $(\exists y)$ " and then " $y \neq x$ ", to the C_k and the assertion, (b), relativizes the A_i to both the C_k and the B_j . To account for sentences like (162) we take the B_j to be vacuous and we replace the comma in (163) with a semi-colon.

To get the semantic analysis of (164) we first write it in the form (180) as

- (182) $(\text{The } x)(\text{Theorem}(x); \text{Fermat-discovered}(x);$
 $\text{Proven}(x))$

in which $n_1 = n_2 = n_3 = 1$ and $C_1 = \text{Theorem}$, $B_1 = \text{Fermat-discovered}$, and $A_1 = \text{Proven}$. This gives us

- (183) (a) $(\exists x)(\text{Theorem}(x); \text{Fermat-discovered}(x),$
 $-(\exists y)(\text{Theorem}(y); \text{Fermat-discovered}(y),$
 $y \neq x))$
- (b) $(\forall x)(\text{Theorem}(x), \text{Fermat-discovered}(x);$
 $\text{Proven}(x))$

as the presupposition and assertion, respectively, of (182) and thus of (164). The meaning of (164) is given by (183) and the meanings of (183a) and (183b), in turn, are given, respectively, by (176) and (171).

We can get the meaning of (178), similarly, first by giving it the semantic representation

(184) (The x) (Theorem(x); Fermat-discovered(x), Known(x);
Proven(x), Published(x))

According to (181), the semantic analysis of (184) is provided by the presupposition and assertion, respectively,

(185) (a) $(\exists x)(\text{Theorem}(x); \text{Fermat-discovered}(x),$
 $\text{Known}(x), \neg(\exists y)(\text{Theorem}(y); \text{Fermat-discovered}(y), \text{Known}(y), y \neq x))$

(b) $(\forall x)(\text{Theorem}(x), \text{Fermat-discovered}(x),$
 $\text{Known}(x); \text{Proven}(x), \text{Published}(x))$.

According to analysis (185), formula (184) presupposes that there is exactly one known theorem discovered by Fermat and asserts that whatever theorems were discovered by Fermat have been proven and published. This is exactly the intuitive meaning of (178).

1.2: Comparative Quantifiers

We can use the framework developed in the preceding section to construct semantic analyses for an entirely new kind of construction, the comparative quantifiers. Each of the following sentences provides some answer to the questions of *How many?* theorems or theories are provable or testable and so can be considered as involving quantifiers:

(186) *More theorems are provable than theories.*

(187) *More theorems than theories are provable.*

(188) *More theories are testable than provable.*

(189) *More theories are testable than theorems are provable.*

Sentences (186) and (187) are clearly synonymous and so should have the same semantic representations. The other two have different meanings and require different semantic representations. Because of the intuitive similarity of *more* to *most* and *more than m/n* we will formulate our analysis along Rescher's and Kaplan's lines, rather than van Fraassen's.

Each of (187) and (188) involves three wffs, so we are tempted to give them semantic representations like

(190) $(\text{More } x)(\text{Theorem}(x), \text{Theory}(x), \text{Provable}(x))$

(191) $(\text{More } x)(\text{Theory}(x), \text{Testable}(x), \text{Provable}(x))$

respectively. When we examine the intended meanings of these formulas, however, an important difference emerges. Formula (190) makes a comparison between its first and second wffs with respect to the third. It says, assuming that it represents (187), that there are more theorems than theories that are provable. Formula (191), in contrast, makes a comparison between its second and third wffs with respect to the first. It says, assuming that it represents (188), that more theories are testable than provable.

It seems, at first, to follow from these facts that there are two different more quantifiers, one that compares the first two of three wffs with respect to the third and one that compares the second and third of three wffs with respect to the first. If we denote these, respectively, by " More_{12} " and " More_{23} ", then we can replace the proposed semantic representations (190) and (191) of sentences (187) and (188), respectively, with the more precise semantic representations

(192) $(\text{More}_{12} x)(\text{Theorem}(x), \text{Theory}(x), \text{Provable}(x))$

(193) $(\text{More}_{23} x)(\text{Theory}(x), \text{Testable}(x), \text{Provable}(x))$.

These formulas express the difference between (187) and (188) that we just discussed by distinguishing two different *more* quantifiers in the two sentences.

Formulas (192) and (193) have a certain plausibility as semantic representations of (187) and (188), respectively, but there are two serious problems with them. In the first place, our solution is ad hoc. Our subscripting device, in these formulas, has no more principled basis than would the use of two entirely different symbols for the two quantifiers in (192) and (193). We might just as well call one "Q" and the other "P" as call them " More_{12} " and " More_{23} ". In the second place, there seems intuitively to be no real difference between the quantifiers in (187) and (188) at all. Although the quantifier works differently in the two sentence, it is difficult to see two really different meanings in the quantifiers of the two sentences.

One possible solution to this problem would be to assume that there is a single *more* quantifier, but that its effect in a particular sentence depends on the semantic character of the wffs that make up the sentence. "*More*" in (190) compares "*Theorem*" to "*Theory*", on this analysis, because it makes more sense to compare theorems to theories with respect to provability than it does to compare theories to provable things with respect to theoremhood. "*More*" in (191) compares "*Testable*" to "*Provable*", similarly, because it makes more sense to compare theories with respect to testability and provability than it does to compare theories and testable things with respect to provability. This solution seems plausible enough, but there is serious question as to whether it could be made precise in formal terms. Even if formalization of this proposal turns out to be

possible, moreover, it would clearly involve a very complicated system of cross-referencing of predicates in the lexicon plus supplementary recursive rules for other wffs to enable us to determine which wffs it makes more sense to compare than which others. This leads us to look, at least, for a simpler solution to the problem of the different interpretations of *more*.

It turns out that a very simple solution to this problem can be formulated in terms of the semi-colon device that we introduced in the last section. If we look for a moment at (189), rather than (187) and (188), we see that there are really two kinds of wffs involved in these comparisons, in the sense of two classes of wffs, each functioning differently in determining the meaning of the sentence. This is just what we saw in the case of the generalized universal and existential quantifiers in Section 1.1. What we are comparing in (189) are testable theories and provable theorems. We can express this fact formally by grouping "Theory" with "Testable" and "Theorem" with "Provable" and separating the two groupings with a semi-colon. This gives us the formula

$$(194) \text{ (More } x)(\text{Theory}(x), \text{Testable}(x); \text{Theorem}(x), \text{Provable}(x))$$

as the semantic representation of (189).

The most important characteristic of (194) is that it is capable of being completely generalized. We can admit semantic representations of the general form

$$(195) \text{ (More } x)(B_1, \dots, B_n; A_1, \dots, A_m)$$

in perfect analogy with the generalized universal and existential quantifiers in (171) and (176), respectively. To give meaning to such formulas, so they can, in fact, serve as semantic representations, we also give them the following semantic analysis:

$$(196) \text{ } f \text{ satisfies (More } \alpha)(B_1, \dots, B_n; A_1, \dots, A_m) \text{ in } \langle DR \rangle \\ \text{if and only if} \\ K(E[x \in D \text{ and } f_x^\alpha \text{ satisfies } B_i \text{ in } \langle DR \rangle, i=1, \dots, n]) \\ > K(E[x \in D \text{ and } f_x^\alpha \text{ satisfies } A_j \text{ in } \langle DR \rangle, j=1, \dots, m])$$

Comparing this formula to formula (39), we see that simple *most* is just the special case of *more* in which $n=m=1$ and $A_1=B_1$.

Comparing (196) to (43) shows us that relativized *most* is the special case of *more* in which $n=m=2$, $A_1=B_1$, and $A_2=B_2$.

Rule (196) uses "x" as the variable that indicates members of the extension of the formula in the scope of "E", so we will have to use a different variable in semantic representations. With this proviso, the rule clearly provides us with the correct semantic analysis of (189), because it tells us that (194), with some other variable in place of "x", is true if and only if the number of testable theories is greater than the number of provable theorems. It also gives us the correct semantic analysis of (187) and (188), if we represent these sentences, respectively, by the formulas

$$(197) \text{ (More } y)(\text{Theorem}(y), \text{Provable}(y); \text{Theory}(y), \text{Provable}(y))$$

$$(198) \text{ (More } y)(\text{Theory}(y), \text{Testable}(y); \text{Theory}(y), \text{Provable}(y))$$

Rule (196) says that (197) is true if and only if there are more provable theorems than provable theories and that (198) is true if and only if there are more testable theories than provable theories. This accords exactly with the intuitive meanings of those sentences.

As we have noted, (196) also tells us that the sentences

Most theories are testable.

More theories are testable than not testable.

are synonymous, as we would expect from a correct analysis of *more*. Most important, however, is the fact that (196) gives us the semantic analyses of kinds of sentences other than those for which we constructed it. We can account for the meaning of a sentence like

$$(199) \text{ More testable theories have been proposed than provable theorems have been published,}$$

for example, by simply taking $n=m=3$ and $B_1=\text{Theory}$, $B_2=\text{Testable}$, $B_3=\text{Proposed}$, $A_1=\text{Theorem}$, $A_2=\text{Provable}$, and $A_3=\text{Published}$. This gives us the semantic representation

$$(200) \text{ (More } y)(\text{Theory}(y), \text{Testable}(y), \text{Proposed}(y); \text{Theorem}(y), \text{Provable}(y), \text{Published}(y)),$$

whose meaning is given automatically by (196). Rule (196) says that (200), and thus (199), is true if and only if the number of testable theories that have been proposed is greater than the number of provable theorems that have been published. As we have seen before, the key to interpreting such formulas is the placement of the semi-colon, that is, the division of the wffs into two differently functioning classes. The fact that (196)

accounts for the meanings of sentences other than those that were considered in arriving at it is strong evidence for the correctness of both the rule and the framework that underlies it.

Section 2: Presupposition and Truth-Value

2.1 Truth-Values and Untruth-Values

We have examined relativization and found that it can be generalized to account for sentences that involve more than two wffs in their meanings. A similar generalization can be developed for the notion of presupposition. We can begin by examining the presuppositional analyses that we gave in Chapter 3.

The first thing we noted about presupposition in 11,3,1 was that it makes sense only if our logic allows more than two truth-values. Each of the analyses we later constructed consisted of two conditions, which were combined in the analyses to account for three truth-values for the sentences analyzed. If we introduce a little redundancy into our analyses, we can reformulate them in such a way that there are three conditions, one for each truth-value.

The sentence

(201) *Only logicians can prove Gödel's theorem.*

for example, has the semantic representation

(202) $(\text{Only } x)(\text{Logician}(x); \text{Prove-Gödel}(x))$

where the semi-colon has now been introduced in the obvious place. The meaning of (202) is given by the presupposition and assertion, respectively,

(203) (a) $(\exists x)(\text{Logician}(x); \text{Prove-Gödel}(x))$

(b) $(\forall x)(\text{Prove-Gödel}(x); \text{Logician}(x)).$

Formula (202), and therefore sentence (201), is true, if both (a) and (b) hold; false, if (a) holds and (b) does not; and third-valued, otherwise.

We can reformulate (203) in terms of three conditions by explicitly writing out the formulas that correspond individually to the three truth-values, as follows:

(204) (a) $(\exists x)(\text{Logician}(x); \text{Prove-Gödel}(x))$
 $\wedge (\forall x)(\text{Prove-Gödel}(x); \text{Logician}(x))$

(204) (b) $(\exists x)(\text{Logician}(x); \text{Prove-Gödel}(x))$
 $\wedge \neg (\forall x)(\text{Prove-Gödel}(x); \text{Logician}(x))$

(c) $\neg (\exists x)(\text{Logician}(x); \text{Prove-Gödel}(x)).$

Formula (204a) is true, if both (203a) and (203b) hold. Formula (204b) is true, if (203a) holds but (203b) does not. Formula (204c) is true, otherwise. It follows that (202) is true, if (204a) is true; false, if (204b) is true; and third-valued, if (204c) is true. It is not difficult to show that one (and only one) of these cases must hold.

Ordinarily we try to eliminate redundancy in linguistic analyses, but in this case we have found it useful to introduce some. Reformulating (203) as (204) has enabled us to establish a one-to-one correspondence between the truth-conditions of our analysis and the truth-values of our logic. It might not be clear how to generalize (203) with respect to truth-value, but generalizing (204) is simply a matter of introducing more truth-values and assigning one truth condition, like those of (204), to a given sentence for each of those truth-values. If the meaning of a sentence S involves n truth-values, its analysis will consist of n statements of the form

(205) S has truth-value i under conditions C , if S_i is true under C

either for $i=1, \dots, n$ or for $i=0, \dots, n-1$, depending on the numbering system we choose. If we denote the third value by 0, as Keenan does, and truth and falsity by 2 and 1, respectively, then a sentence S is presuppositional, in the sense we discussed in Chapter 3, if there are two sentences S_a and S_b such that $S_0 = \neg S_a$, $S_1 = S_a \wedge \neg S_b$, and $S_2 = S_a \wedge S_b$.

Multiple truth-values have been interpreted in various ways in the logical literature, a particularly interesting one being the tense-logic of Prior (1956), for example. An intuitively natural interpretation of multiple truth-values as a generalization of presupposition, which is what we are interested in, can be obtained by reversing the numbering system suggested by Keenan. Instead of using 0 to denote "neither true nor false", we can denote truth by 0, falsity by 1, and other truth-values by the other positive integers in sequence. No matter what kind of sentences we are dealing with we will always need truth and falsity, 0 and 1, and we can get as many additional truth-values as we need by simply taking more integers in order.

With this numbering system it becomes more natural to view "truth-values" as "untruth-values", denoting "degrees of

untruth" or "modes of untruth", that is, ways in which a sentence can fail to be true. Keenan himself says that "The effect of the third values is to distinguish two ways a sentence can be untrue" (1971a, p. 276), but for some reason he chooses to denote that "third" value by "zero", rather than trying to develop a principled numbering system.

In the system suggested here a true sentence, since it is not untrue at all, would naturally have an untruth-value of 0. A false sentence is untrue in the first possible way, namely, the failure of its truth condition or assertion, so it is natural to give it the untruth-value 1. A presuppositional sentence whose presupposition fails to hold is untrue in the second possible way, so it is natural to give such a sentence the untruth-value 2. As we introduce new ways in which a sentence can fail to be true, we can keep track of them by introducing additional untruth-values. It follows that we replace (205) in semantic analyses with

(206) S has untruth-value i under conditions C, if S_i is true under C

for $i=0, \dots, n-1$. In our discussion of presuppositional sentences just after (205) we must interchange S_0 and S_2 to get the correct account of presupposition. We will not, in general, discuss multiple untruth-values explicitly after the next section, but whatever we say in III or IV about sentences can be applied equally well to the truth-conditional formulas S_i of sentences with more than two.

2.2: Multiple Untruth-Values

We have seen that "ordinary" sentences involve two untruth-values and that the semantic analyses of sentences with presuppositions require three. To get a clearer idea of what it might mean for a sentence to involve four untruth-values we can examine a sentence like

(207) *Only the author of Syntactic Structures really believed it.*

which contains the two presuppositional quantifiers *only* and *the*.

Sentence (207) differs considerably from the superficially similar sentence

(208) *Only authors of Syntactic Structures really believed it.*

which is analyzable in terms of (100). The semantic representation of (208) is

(209) $(\text{Only } x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

the presupposition and assertion of which are, respectively,

(210) (a) $(\exists x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

(b) $(\forall x)(\text{Believed-SS}(x); \text{Wrote-SS}(x))$

Formula (209), and thus sentence (208), has untruth-value 0, if both (a) and (b) of (210) hold; 1, if (a) holds but (b) does not; and 2, otherwise. In terms of (206) we get

(211) $S_0 = (\exists x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$
 $\quad \quad \quad \wedge (\forall x)(\text{Believed-SS}(x); \text{Wrote-SS}(x))$

$S_1 = (\exists x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$
 $\quad \quad \quad (\forall x)(\text{Believed-SS}(x); \text{Wrote-SS}(x))$

$S_2 = -(\exists x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

as the semantic analysis of (209) and (208).

Sentence (207) also differs considerably from the superficially similar sentence

(212) *The author of Syntactic Structures really believed it.*

The semantic representation of (212) is the formula

(213) $(\text{The } x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

and its presupposition and assertion, respectively, are

(214) (a) $(\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x))$

(b) $(\exists x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

where we have taken the B_i of (181) to be vacuous, as noted in connection with (181) and (162). Formula (213), and thus sentence (212), has untruth-value 0, if both (a) and (b) of (214) hold; 1, if (a) holds but (b) does not; and 2, otherwise. In terms of (206) we get

(215) $S_0 = (\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x))$
 $\quad \quad \quad \wedge (\forall x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

$S_1 = (\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x))$
 $\quad \quad \quad \wedge -(\forall x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$

$$(215) S_2 = -(\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x))$$

as the semantic analysis of (213) and (212).

As we saw in 11,3,2.2, the semantic effect of *only* is to presuppose the sentence to which it is added and to limit the extension of the main wff of that sentence to its relativization wff. Both intuitively and from that analysis we see that (207) presupposes (212). If it is not the case that the author of *Syntactic Structures* really believed it, then he cannot be the *only* one who really believed. This would be quite a different state of the world from the one in which the author of *Syntactic Structures* really believed it, but someone else really believed it, too. In the latter case (207) would be false. He believed it, but he was not the only one who did. In the former case, however, the question of his being the only one does not even arise, because he is not a believer himself. Sentence (212) must be true in order for (207) to be either true or false, that is, in order for the question that (207) answers to come up at all. If (212) is true, that is, if the author of *Syntactic Structures* really believed it, then (207) is false or true, respectively, according as there was or was not someone else who believed it. It follows both that (207) presupposes (212) or, equivalently, (213) and that it asserts (210b), the assertion of (208) and (209).

To say that (207) presupposes (212) is to say that (212) must be true in order for (207) to be either true or false. If (212) is not true, then (207) must have a truth-value other than truth or falsity, that is, an untruth-value other than 0 or 1. This automatically requires the introduction of a third truth-value or untruth-value to account for the semantics of (207). Sentence (212), however, can itself be untrue in two distinct ways. Since it is itself presuppositional, we already need to recognize an untruth-value other than 0 or 1 to account for its semantics, before we even get to consider (207). Sentence (212) will have this other untruth-value, if its own presupposition, formula (214a), is false.

This gives us the following situation for (207). Sentence (207) is true, if (212) is true and (210b) is true. Sentence (207) is false, if (212) is true but (210b) is false. If (212) is untrue, then (207) is neither true nor false, but has an untruth-value different from 0 or 1. Sentence (212), however, can be untrue in either of two quite distinct ways. It can have untruth-value 1, that is, it can be false, simply because its assertion (214b) is false, while its presupposition is (214a) is true. It can also have untruth-value 2, however, if its presupposition (214a) is false. We see that there are two very different ways in which the presupposition of (207) can fail, that is, two very different ways in which the question answered

by (207) can fail to arise, or, in still other terms, two very different ways in which (207) can fail to be either true or false.

Sentence (207) can fail to be either true or false either because its presupposition is false or because its presupposition is neither true nor false. These are clearly two distinct ways in which (207) can fail to be true and so must express two distinct untruth-values, by the very definition of untruth-value. It follows that we need two, not just one, untruth-values different from 0 and 1 to account for the semantics of (207). As usual, 0 expresses the case in which both the presupposition and the assertion are true, 1 expresses the case in which the presupposition is true but the assertion is false, and 2 expresses the case in which the presupposition is false. The new untruth-value 3 expresses the new case in which the presupposition fails to be either true or false, because its own presupposition is false.

Explicitly in terms of (206) this gives us the untruth conditions

$$(216) S_0 = (\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x)) \\ \wedge (\forall x)(\text{Wrote-SS}(x); \text{Believed-SS}(x)) \\ \wedge (\forall x)(\text{Believed-SS}(x); \text{Wrote-SS}(x))$$

$$S_1 = (\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x)) \\ \wedge (\forall x)(\text{Wrote-SS}(x); \text{Believed-SS}(x)) \\ \wedge -(\forall x)(\text{Believed-SS}(x); \text{Wrote-SS}(x))$$

$$S_2 = (\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x)) \\ \wedge -(\forall x)(\text{Wrote-SS}(x); \text{Believed-SS}(x))$$

$$S_3 = -(\exists x)(\text{Wrote-SS}(x); -(\exists y)(\text{Wrote-SS}(y); y \neq x))$$

as the semantic analysis of (207). Formula (216) S_0 is the conjunction of (215) S_0 and (210b). Formula (216) S_1 is the conjunction of (215) S_0 and the negation of (210b). Formula (216) S_2 is the same as formula (215) S_1 and formula (216) S_3 is the same as formula (215) S_2 . Each of these formulations is derived from our foregoing discussion of sentence (207) and new untruth-values for more complicated sentences can be introduced in a similar way.

Quantifiers as Logical Functions

CHAPTER I: THE BINDING PROPERTY

Section 1: Quantifiers and Propositional Functions

As was first recognized by Mostowski (1957), the key to defining a general notion of quantifier is "the elementary fact that quantifiers enable us to construct propositions from propositional functions with one argument" (p. 13). Intuitively a propositional function is a mapping from n -tuples of individuals to propositions and is denoted by an open wff with n free variables. Given a sequence of individuals, a propositional function "says something" about those individuals and we can determine what it says by taking those individuals and the values of the variables in the wff that represents the propositional function. A propositional function of no arguments, represented by a closed wff, which has no free variables, amounts to a proposition, because what it says is independent of what values are assigned to which variables.

Like other logicians, Mostowski formalizes the notion of propositional function, in essence, as a mapping from n -tuples of individuals to truth-values. A propositional function assigns to each sequence of individuals one of the truth-values "truth" or "falsity", without regard to any notion of proposition. This formalization is less intuitive than talking explicitly about propositions, but it has the advantage, for philosophers, of avoiding the ontological problem of determining what "kind" of entity propositions are. Our purely linguistic purposes do not require us to worry about this problem, however, any more than the atomic physicist has to worry about what "kind" of entity, in the ontological sense, his μ -mesons, neutrinos, and quarks are. We can therefore adopt the more intuitive interpretation with a clear conscience. A principled philosophical justification for this decision can also be formulated, more or less along the lines of Putnam (1971).

As Stolyar (1970) explains,

The operation of binding with a quantifier or, as it is called, quantification, converts a one-place predicate into a proposition. To convert a many-place predicate into a proposition by means of quantification, we need to put as many quantifiers in front of it as there are distinct variables, thus binding each individual variable by means of a quantifier. (p. 152)

As illustration, Stolyar gives eight examples of universal and existential quantification, as follows:

1. $(\forall x)(\forall y)R(x,y)$ "for every x and every y , the proposition $R(x,y)$ holds";
2. $(\forall y)(\forall x)R(x,y)$ "for every y and every x , the proposition $R(x,y)$ holds";
3. $(\forall x)(\exists y)R(x,y)$ "for every x , there exists a y such that $R(x,y)$ holds";
4. $(\exists y)(\forall x)R(x,y)$ "there exists a y such that, for every x , the proposition $R(x,y)$ holds";
5. $(\exists x)(\forall y)R(x,y)$ "there exists an x such that, for every y , the proposition $R(x,y)$ holds";
6. $(\forall y)(\exists x)R(x,y)$ "for every y , there exists an x such that $R(x,y)$ holds";
7. $(\exists x)(\exists y)R(x,y)$ "there exists an x and there exists a y such that $R(x,y)$ holds";
8. $(\exists y)(\exists x)R(x,y)$ "there exists a y and there exists an x such that $R(x,y)$ holds."

These formulas constitute all possible universal and existential quantifications of a binary predicate, that is, of an open wff of two free variables, which represents a propositional function of two arguments.

Stolyar goes on to explain that, just as a quantifier turns a one-place predicate (open wff with one free variable) into a proposition (closed wff), it is also the case that

If a two-place predicate is bound by a single quantifier, for example, $(\forall x)R(x,y)$, then the resulting formula expresses not a proposition, but a logical function of the second variable, which is not bound by the quantifier (and we say that it is a free variable). This formula is a one-place predicate.

To illustrate this point, Stolyar gives the following examples:

Suppose, for example, that x and y are variables for real numbers (in which case we usually say "x and y are real numbers") and suppose that $<$ is the symbol for the two-place predicate (relation) that we know as "less than." Then

$(\forall x)(\forall y)[x < y]$ and $(\forall y)(\forall x)[x < y]$	are false propositions
$(\forall x)(\exists y)[x < y]$	is a true proposition, but
$(\exists y)(\forall x)[x < y]$	is a false proposition;
$(\exists x)(\forall y)[x < y]$	is a false proposition, but
$(\forall y)(\exists x)[x < y]$	is a true proposition;
$(\exists x)(\exists y)[x < y]$ and $(\exists y)(\exists x)[x < y]$	are true propositions;
$(\forall x)[x < y]$	is a logical function of y ;
$(\exists y)[x < y]$	is a logical function of x .

In each case the first (right-most) quantifier turns the binary predicate into a unary predicate and the second quantifier, if there is one, turns that unary predicate into a "sentence", that is, a formula representing a proposition.

We can generalize Stolyar's observations to say that quantifiers map m -argument propositional functions onto $m-1$ -argument propositional functions. From the three-argument propositional function

$$x+y=z$$

for example, we can get each of the two-argument propositional functions

$$(\forall x)(x+y=z)$$

$$(\forall y)(x+y=z)$$

$$(\forall z)(x+y=z)$$

through universal quantification of the respective variables. For the same three-argument propositional function we can also get each of the three existential quantifications

$$(\exists x)(x+y=z)$$

$$(\exists y)(x+y=z)$$

$$(\exists z)(x+y=z)$$

each of which is a propositional function of two arguments, and we can do exactly the same thing with any other quantifier as well.

Most generally, in accordance with our discussion in 11,4, we can say that quantifiers map n -tuples of m -argument propositional functions onto $m-1$ -argument propositional functions. We can universally quantify the ordered triple of two-argument functions

$$(1) (\text{Theorem}(x), \text{Logician}(y), \text{Provable}(x,y))$$

for example, to get the single one-argument function

$$(2) (\text{All } y)(\text{Theorem}(x), \text{Logician}(y); \text{Provable}(x,y))$$

and then (2), in turn, can be existentially quantified to produce the zero-argument function or proposition represented by

$$(3) (\text{Some } x)(\text{All } y)(\text{Theorem}(x), \text{Logician}(y); \text{Provable}(x,y)).$$

The wffs "Theorem(x)" and "Logician(y)" in these formulas explicitly contain only one free variable each, but they can always be considered as functions of two arguments, in which the second plays no role. The presence of "Provable(x,y)" in (1) automatically makes (1) a function of two arguments, so it makes sense to use the two-argument interpretation of the other two wffs in (1).

Formula (1) represents the relation of provability that can hold between theorems and logicians. Formula (2) represents the property that theorems can have of being provable by all logicians. Formula (3) represents the proposition that some

theorems have the property (2), that is, that some theorems can be proven by all logicians. *All*, plus the indicated placement of the semi-colon, turns the triple of propositional functions in (1) into the single propositional function (2). A different placement of the semi-colon would produce a different result, as would the quantification by *all* of a different variable. *Some* then turns the propositional function (2) into the different one (3), which has one fewer free variable. In each case the quantifier reduces the total number of arguments in the propositional functions involved by one.

To make this point even clearer, we can easily construct an example in which each wff has a full complement of free variables, in contrast to the situation we noted in connection with the last example. Let us examine the ordered pair of two-argument propositional functions

(4) (Older(x,y), Bigger(x,y))

and the ordered triple of two-argument propositional functions

(5) (Hotter(x,y), Thicker(x,y), Next-to(x,y)).

Through universal quantification we can turn (4) into the single one-argument propositional function

(6) (All y) (Older(x,y); Bigger(x,y))

which represents the property of being bigger than everything younger, and through existential quantification we can turn

(5) into the single one-argument propositional function

(7) (Some y) (Hotter(x,y), Thicker(x,y); Next-to(x,y))

which represents the property of being next to something cooler and thinner. *Many*-quantification can then be applied to turn the ordered pair of one-argument propositional functions

((6); (7))

into

(8) (Many x) ((All y) (Older(x,y); Bigger(x,y)); (Some y) (Hotter(x,y), Thicker(x,y); Next-to(x,y)))

which represents a single zero-argument propositional function, that is, a proposition.

Formula (8) represents the proposition that many things that are bigger than everything they are older than are next to some things they are hotter and thicker than. This is a peculiar proposition because of its complexity and the low likelihood

that anyone would ever want to assert it, but it is perfectly acceptable from both the syntactic and semantic points of view. With a little more ingenuity one could undoubtedly come up with a more natural example, but (8) suffices for our purposes. Its meaning is obtainable directly from the semantic analyses of the quantifiers it contains that we gave in 11.

The most important thing to notice in our discussion of (8) is that the number *n* of predicates, wffs, or propositional functions is entirely independent of the number *m* of free variables or arguments that they contain. The effect of the quantifier in each of the cases (6), (7), and (8) is to reduce *m* to *m*-1 and *n* to 1. Quantifiers map *n*-tuples of propositional functions onto propositional functions and they reduce the total number of arguments in those functions by one.

It follows that quantificational sentences can always be given semantic representations of the form

(9) $(Q_1 x_1) \dots (Q_m x_m) (A_1(x_1, \dots, x_m), \dots, A_n(x_1, \dots, x_m))$

in which the Q_i are quantifiers and in which some of the commas between successive A_j 's may be replaced by semi-colons in

specific cases. It is worth noting that we arrived at the semantic representation schema (9) and the generalizations it incorporates by eschewing the usual ontological scruples about propositions and talking freely in terms of propositional functions as mappings from *n*-tuples of individuals onto propositions. We might have missed these generalizations if we had restricted ourselves to Mostowski's less intuitive formal characterization of propositional functions as mappings from *n*-tuples of individuals to truth-values.

Section 2: Keenan's Conception of Binding

Keenan does not give an explicit definition of the property incorporated in (9), but only a vague discussion of "the intuition behind it." He expresses the peculiar idea that binding has something to do with conjunction, more specifically, that it arises from the interaction of quantifiers with other operators like conjunction. Keenan gives the formulas

(10) $(C(j) \ \& \ L(j))$

(11) $(\exists x)(C(x) \ \& \ L(x))$

and observes that, whereas in (10) we can determine the truth or falsity of the two conjuncts " $C(j)$ " and " $L(j)$ " independently of each other, we cannot do the same in the case of " $C(x)$ " and " $L(x)$ " in (11). "What were independently meaningful sentential

parts of the operand sentence have become bound together in the resultant sentence" (P. 265).

Keenan's observation is, of course, correct, but it misses the point as an explanation of binding. As traditional logical terminology suggests, it is primarily the variable, "x", that is bound in (11), not the conjuncts "C(x)" and "L(x)". The quantifier "∃", meaning *some*, takes the free variable "x" and turns it into a bound variable. This means that what is a variable in

$$(12) C(x) \& L(x)$$

in the sense that its value can "vary" by having various individuals substitute for it to produce sentences like (10), is no longer a variable, in that sense, in (11). The "x" in (12) is "free" to take individuals as values, but the "x" in (11) cannot do this, because it is "bound" to the quantifier. We have already seen in 11.3.2.1 that Keenan has difficulty with the distinction between free and bound variables and tries to overcome this difficulty by introducing a new distinction between names and "arbitrary" names, which is subject to exactly the same objection as Keenan raises to the distinction between free and bound variables. Keenan's misconceptions about binding undoubtedly stem from his confusion about this distinction.

That binding has nothing at all to do with conjunction can be seen even more clearly from the formula

$$(\exists x)C(x)$$

in which the "x" is just as tightly bound to the quantifier as it is in (11), but in which there is no other conjunct, like the "L(x)" in (11), that is "bound" to "C(x)". Both "x" and "y" are free in (4) and (5), as a further example, but "x" is bound in (8) and "y" is bound in (6), (7), and (8), even though neither (6), (7), nor (8) contains any conjunction (except indirectly through their simple reductions).

The reason Keenan's conjuncts are "bound" to each other, in his sense, is, in fact, precisely that the variable they contain is bound to the quantifier, in the usual sense. Bound variables are more like co-occurrence indices than what one usually thinks of as variables. Their main function is to tell us which argument in a propositional function the quantifier they occur with is eliminating, when it reduces m to $m-1$. Mathematicians often use the term "dummy variable" in a similar way, to refer to symbols that look like variables, but are really not, because they lack the freedom to vary. It is worth noting, in passing, that the sense of variable which Keenan tries to replace, by changing free variables to "arbitrary names", is the sense in which variables are really variables, while the sense which he keeps, namely, bound variables, lacks the basic property of variability that makes real variables variables. This

This reversal is just another reflection of his fundamental confusion on logical notation.

Section 3: Vacuous Quantification

There seems to be general agreement among logicians that vacuous quantification, at least in the universal and existential cases, coincides with the identity function. Stolyar points out, for example, that the typical universal and existential quantifications can be written in the general forms

$$(13) (\forall x_i)\phi(\dots x_i \dots)$$

$$(14) (\exists x_i)\phi(\dots x_i \dots)$$

respectively, but he goes on to say that

The requirement that x_i appear free in ϕ is not obligatory: we may assume that, for a predicate formula ϕ that is independent of x , $(\forall x)\phi$ and $(\exists x)\phi$ simply coincide with ϕ . (p. 157)

Church adds a footnote to his semantic rule for *all* to the effect that

If \underline{A} does not contain the individual variable \underline{a} as a free variable, the value of $(\forall \underline{a})\underline{A}$ is the same as the value of \underline{A} , for any system of values of the free variables. If \underline{A} contains no free variables, and if \underline{a} is any individual variable, $(\forall \underline{a})\underline{A}$ has the same denotation as \underline{A} . (P. 176, n. 313)

Carnap (1958) also adopts the convention that a universally or existentially quantified formula has the same truth-value as the wff being quantified, if the variable of the quantifier does not occur free in the wff.

Keenan, however, is as confused on this question as he is about binding in general. He gives us the formulas

$$(15) (\forall x)C(j)$$

$$(16) (\forall x)(\exists x)(C(x) \& L(x))$$

and tells us that

These strings are intuitively meaningless because they purport to be saying something about everything but do not. Analogous strings in English would be: *For all x, John came early* and *For all x, someone came early and left late*. Thus we do not count them as legitimate meaning representations.... (p. 261)

As we saw in 11,1,1, however, formulas (15) and (16) are not "intuitively meaningless" at all. According to the semantic analyses we saw in that section, a formula of the form

$$(17) (\forall x)A$$

says that "A" is true no matter what individual we assign as a value to "x". Formula (15), therefore, says that "C(j)" is true no matter what individual we assign as a value to "x". Is this "intuitively meaningless"? Is it "intuitively meaningless" to say that (16) is true under an assignment function \underline{d} if and only if the formula

$$(18) (\exists x)(C(x) \ \& \ L(x))$$

is true under every assignment function \underline{d}' which is like \underline{d} except, perhaps, at "x", that is, if and only if (18) is true no matter what value we assign to "x"? The fact is that (15) and (16) are "saying something about everything," because, as we saw in 11,1,1, to "say something about everything" is to say that "something" is true no matter what value we assign to some variable, whether or not that variable occurs free in the formula that that "something" is represented by.

It does make sense, in other words, to talk about a formula being true for all values of variables it does not contain. To rule out such talk would significantly complicate the definitions of truth and satisfaction by permitting quantifiers to be partial functions, rather than requiring them to assign a value to every propositional function. In Quine's words,

Suppose \underline{x} is a function, or one-many relation, which assigns an entity to each variable. In Tarski's terminology, \underline{x} is said to satisfy a formula \underline{y} of \underline{L} if \underline{y} comes out true for the values of its free variables which are assigned to those variables by \underline{x} . Vacuously, then, if \underline{y} is a statement (hence devoid of free variables), \underline{y} is satisfied by every function \underline{x} or by none according as \underline{y} is true or false.

(p. 142)

We can say a formula is true for all values of a variable it does not contain, if we can say it is true without mentioning that variable. We can say that

$$(19) \text{Provable}(x_1, x_2, x_3)$$

for example, is true for certain values of " x_1 ", " x_2 ", and " x_3 ", so we can say it is true for those values of " x_1 ", " x_2 ", and " x_3 ".

and for all values of " x_5 ", because it does not matter what the value of " x_5 " is. This, however, is the same as saying that the formula

$$(20) (\text{All } x_5) \text{Provable}(x_1, x_2, x_3)$$

is true for the specified values of " x_1 ", " x_2 ", and " x_3 ". It cannot be "intuitively meaningless", if it is true.

We can consider it established that vacuous quantification is both intuitively meaningful and formally motivated. When we try to generalize it beyond the universal and existential cases, however, we discover that its generally accepted identification with the identity function must be considerably weakened. We cannot say, in general, for any quantifier, that a quantifier leaves a formula unchanged, if the formula does not contain the variable of quantification as a free variable. We cannot even say, in fact, that the truth-value of the formula is left unchanged in such a case. To say that a formula is true for no values of a variable, for example, is to say that its negation is true for all values of that variable, because of interdefinability. If we say that (20) is true for the same values of " x_1 ", " x_2 ", and " x_3 " as is (19), in other words, because (19) does not contain " x_5 ", then consistency requires us to say that

$$(21) (\text{No } x_5) \text{Provable}(x_1, x_2, x_3)$$

is false for exactly those values of " x_1 ", " x_2 ", and " x_3 " for the same reason. This follows from the requirement that (21) must be equivalent to

$$(22) (\text{All } x_5) \neg \text{Provable}(x_1, x_2, x_3)$$

which is false, whenever (20) is true.

A no-quantification has the truth-value opposite to that of the wff it quantifies, in contrast to all quantification, which leaves the truth-value of that wff unchanged, when the variable of quantification does not occur free in the wff. The individuals that satisfy a no-quantification are exactly those that fail to satisfy the corresponding all-quantification. It follows that we cannot identify vacuous no-quantification with the identity function, as we can in the case of vacuous all- and some-quantification. One thing that does remain constant in vacuous no-quantification, however, is the number of arguments, that is, free variables, that appear in the formula. Just as non-vacuous quantification reduces the number of arguments by one, as we saw in Section 1, we now see that vacuous quantification leaves that number unchanged. We can replace the more stringent condition of identity with the identity function with this weaker result for vacuous quantification in general.

Section 4: The Formal Definition of Binding

We are now in a position to give a complete and precise formal definition of the binding property. Throughout the rest of this study we will use the symbol " L " to denote a fixed language or set of languages and the symbol " R " to denote the set of propositional functions, also called "relations", that are expressed by the formulas of L . The symbols " R_j " and " R^n ", respectively, will be used to denote the subset of R that consists of j -argument propositional functions (j -place relations) and the n -th Cartesian power of R , that is, the set of n -tuples of members of R . The most interesting case for the linguist is, of course, the one in which L consists of all and only the possible human languages or actual human languages, but we will formulate our results in terms of L in order to maximize generality.

As the formal definition of binding we now get the following formulation:

Definition 1 (The Binding Property): Let F be a mapping

from R^n into R . F is said to be binding in L , if there is an integer i , $1 \leq i$, such that, for all

j_k , $k=1, \dots, n$, the set

$$F\left(\prod_{k=1}^n R_{j_k}\right)$$

is a subset of

(a) R_m , $0 \leq m \leq i$;

(b) R_{m-1} , $i \leq m$;

where $m = \max_{k=1}^n j_k$.

This definition partially characterizes the kind of semantic entity that is expressed by semantic representations of the form (9) and that are interpreted intuitively as quantifiers. Both " m " and " n " have the same meanings in Definition 1 that they have in (9). Condition (b) of the definition expresses the semantic character of binding in non-vacuous quantification, as discussed in Section 1. Condition (a) expresses the semantic character of binding in vacuous quantification, as discussed in Section 3. It says, for example, that "universal quantification of the fifth variable" has no effect on the number of arguments of propositional functions that contain fewer than five arguments, as we saw in connection with (20) and (21).

CHAPTER 2: SET-THEORETIC EXPRESSABILITY

Section 1: Quantifiers and Sets

As we saw in I and II, a quantifier is a semantic operator that answers one of the questions *How many?* or *How much?* The sentence

(23) *Some things are worthwhile.*

for example, answers the question *How many things are worthwhile?* and the sentence

(24) *Many linguistic claims can be proven.*

answers the question *How many linguistic claims can be proven?*

Another way of formulating this intuitive characterization of quantifiers is to say that quantifiers give some indication as to the sizes of sets. Sentence (23) tells us that the set of worthwhile things is not empty and sentence (24) tells us that the set of provable linguistic claims contains at least n members, where n is the manifold size index. In Section 2 we will formalize this relation between quantifiers and sets and in Section 3 we will examine the role that it plays in distinguishing quantifiers from similar, but non-quantificational, operators. In the next chapter we will combine these results with the binding property, which we analyzed in the last chapter, to develop a complete formal semantic answer to the question *What is a quantifier?*

Section 2: Set-Theoretic Relations

The relation between quantifiers and sets is most clearly illustrated by the analysis of *most* that we examined in II,2,2. Kaplan's analysis is formulated explicitly in terms of sets and their cardinalities. It tells us that a formula of the form

(25) *Most α ϕ*

is true under an assignment f of individuals as values to variables if and only if the relation

(26) $K\left(\bigcup_x [x \in D \text{ and } f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle]\right) > K\left(\bigcup_x [x \in D \text{ and } f_x^\alpha \text{ satisfies } \neg\phi \text{ in } \langle DR \rangle]\right)$

holds. The symbol " α " in this analysis is a metalanguage variable that takes object-language variables as values and the symbol " x " is a metalanguage variable that takes individuals in D as values. The object-language is the language under analysis, according to the usual terminology, and the metalanguage is the language in which the analysis is expressed. Both (25) and (26)

are metalanguage formulas, with (25) serving as a schema for object-language formulas. Its instances are obtained by replacing " α " with object-language variables and " ϕ " with object-language wffs.

Formula (26) is a metalanguage sentence that expresses a relation between the cardinalities of the two sets

$$(27) \quad \sum_x [x \in D \text{ and } f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle]$$

$$(28) \quad \sum_x [x \in D \text{ and } f_x^\alpha \text{ satisfies } \neg\phi \text{ in } \langle DR \rangle]$$

Formula (27) denotes the set of individuals in D that make the formula

$$(29) \quad x \in D \text{ and } f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle$$

true, when assigned as values to " x ", and formula (28) denotes the set of individuals in D that make the formula

$$(30) \quad x \in D \text{ and } f_x^\alpha \text{ satisfies } \neg\phi \text{ in } \langle DR \rangle$$

true, when assigned in that way. Since an individual makes (30) true if and only if it makes (29) false, it follows that (28) is the complement of (27).

An individual makes (29) true if and only if it makes each of the formulas

$$(31) \quad x \in D$$

$$(32) \quad f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle$$

true. It follows that an individual belongs to the set (27) if and only if it belongs to both of the sets

$$(33) \quad \sum_x [x \in D]$$

$$(34) \quad \sum_x [f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle].$$

The set (33) is the set of individuals that belong to D , so it is simply D itself. Symbolically, we have

$$(35) \quad \sum_x [x \in D] = D.$$

The set (34) can also be written more compactly, if we introduce a new notation through the following definition:

$$(36) \quad \phi_f^\alpha = \sum_x [f_x^\alpha \text{ satisfies } \phi \text{ in } \langle DR \rangle].$$

Combining (35) and (36), we can say that an individual makes (29) true if and only if it belongs to the set

$$(37) \quad D \cap \phi_f^\alpha$$

which is the intersection of (33) and (34).

In a similar way we can denote the set of individuals that make (30) true by

$$(38) \quad (-\phi)_f^\alpha$$

Since an individual makes (30) true if and only if it makes (29) false, it follows that (38) is the same set as

$$(39) \quad \text{Comp } \phi_f^\alpha$$

the complement of (34), written in accordance with (36).

The equivalence of (38) and (39) gives us

$$(40) \quad D \cap \text{Comp } \phi_f^\alpha$$

as a more compact way of writing (28), just as (37) is a more compact way of writing (27).

Combining (37) and (40), we can now reformulate (26) as

$$(41) \quad K(D \cap \phi_f^\alpha) > K(D \cap \text{Comp } \phi_f^\alpha).$$

Formula (41) is very clearly a relation that involves only sets as arguments and formula (25) is true under f if and only if (41) holds. This follows from the equivalence of (41) and (26). It is a basic property of quantifiers that they can always be expressed in this way, in terms of a truth condition (or set of untruth conditions) that is expressible entirely in terms of set theory.

We can formulate this property most generally as follows

Definition 2 (Set-Theoretic Expressability): Let $\langle DR \rangle$ be a fixed model of L , let F be a mapping from R^n into R , and let g be the set of assignments of individuals in D as values to variables in L . F is said to be expressible in set theory with respect to L and $\langle DR \rangle$ or set-theoretically expressible in L and $\langle DR \rangle$, if there is a function s from $g \times R$ into 2^D and a set-theoretic relation R_F in $(2^D)^{n+1}$ such that, for every A_i in R ,

- (42) f satisfies $F(A_1, \dots, A_n)$ if and only if $R_F(D, s(f, A_1), \dots, s(f, A_n))$.

If F is set-theoretically expressible in L and $\langle DR \rangle$ and (42) holds, then R_F is said to be the set-theoretic relation expressed by F under s in L and $\langle DR \rangle$ and s is said to be the set assignment of F with respect to R_F in L and $\langle DR \rangle$. We also say that F is expressible as R_F under s and that F is expressible under s .

The symbol " 2^D " in this definition, as is customary in set theory, is used to denote the set of subsets of D , because the cardinality of that set is equal to $2^{K(D)}$. As usual, we are interested primarily in the case in which L is the set of possible or actual human languages and $\langle DR \rangle$ is the usual every-day model of those languages, but we express the definition in its full generality.

The fact that quantifiers are always expressible in set theory was first pointed out to me by David Kaplan (personal communication), but the detailed formalization that we just carried out and that we will use later in our full characterization of quantifiers is my own. The fact that the binding property alone does not suffice to characterize quantifiers was first made clear to me by Barbara Partee (personal communication).

In our example, in which we showed that the semantic analysis of (25) can be formulated set-theoretically in the form (41), we have

$$F = \text{Most } \alpha$$

$$n = 1$$

$$s(f, A) = A_f^{\alpha} \text{ for all } A \text{ in } R, f \text{ in } g$$

$$\langle DR \rangle$$

$$R_F(a, b) = (K(a \cap b) > K(a \cap \text{Comp } b)) \text{ for all } a, b \text{ in } 2^D.$$

To account for quantifiers in their full generality, as we discussed this in 11.4, we adopt the convention that, for a quantifier that involves more than two untruth-values, the relation R_F in Definition 2 can be an ordered set of such relations, one for each untruth condition of the form 11(206).

Mostowski (1957) develops a general formal notion of quantifier, but he takes for granted that he is dealing only with set-theoretic operators and so does not bother to discuss set-theoretic expressability explicitly. For our purposes, however, something like Definition 2 is essential, because, as linguists, we are concerned with whatever operators happen to occur in natural languages and we must have a way to distinguish those that are expressible in set theory, in a very precise sense, from those which are not. This will become much clearer in the next section, where we examine two semantic operators which are both binding and which differ only in that one is expressible in set theory, in the sense of Definition 2, while the other is not.

Section 3: Horn's Operators

We can gain an appreciation of the significance of set-theoretic expressability by comparing a couple of similar operators with respect to quantificational status. Horn (1969) points out that *only* acts like a certain class of quantifiers in various ways, but he stops short of saying that it *is* a quantifier. He proposes the following analyses of *only* and the similar operator *even*:

- (43) Only ($x=a$, Fx)

$$P: Fx$$

$$A: \neg(\exists y)(y \neq x \ \& \ Fy)$$

- (44) Even ($x=a$, Fx)

$$P: (\exists y)(y \neq x \ \& \ Fy)$$

$$A: Fx.$$

From the interdefinability of *all* and *some* and the equivalence of the formulas

$$y \neq x \ \& \ Fy$$

$$\neg(y=x) \vee \neg Fy$$

$$\neg(Fy \supset y=x)$$

it follows that (43) is just an earlier notational variant of Keenan's (1971a) analysis of *only*, which we examined in 11,3,2.1.

The set-theoretic expressability of *only* follows from that of *all* and *some*, because Keenan's and Horn's equivalent analyses of *only*, as special cases of our analysis 11(100), are expressible in terms of *all* and *some*. The set-theoretic expressability of *all* and *some* can be demonstrated by finding set-theoretic relations R_F and set assignments s such that (42) holds for each of the two quantifiers.

The presupposition of *only* is of the form

$$(45) \quad (\text{Some } x)(B;A)$$

Formula (45) is true under an assignment f if and only if there is at least one individual that makes "B" true, when assigned as a value to "x", that also makes "A" true, when so assigned. This means that (45) is true if and only if there is at least one individual that makes both "B" and "A" true, that is, if and only if the set of individuals with both the properties "satisfies B" and "satisfies A", which we can abbreviate simply as "B" and "A", respectively, contains at least one member and so is not empty. This gives us

$$(46) \quad D \cap B_F^x \cap A_F^x \neq \lambda$$

where " λ " denotes the empty set, as a set-theoretic formulation of the truth condition of (45). In terms of Definition 2 this gives us

$$F = \text{Some } x$$

$$n = 2$$

$$s(f,A) = A_F^x$$

$$R_F(a,b,c) = (a \cap b \cap c \neq \lambda),$$

as the function, number of wff arguments, set assignment, and set-theoretic relation, respectively.

The assertion of *only*, corresponding to the presupposition (45), is of the form

$$(47) \quad (\text{All } x)(A;B).$$

Formula (47) is true under f if and only if every individual that makes "A" true, when assigned as a value to "x", makes "B" true, when so assigned. This means that (47) is true if and only if

every member of the set of individuals that have the property "A" is a member of the set of individuals that have the property "B", that is, if and only if the set of individuals that have "A" is a subset of the set of individuals that have "B". This gives us

$$(48) \quad D \cap A_F^x \subseteq B_F^x$$

as a set-theoretic formulation of the truth-condition of (47) and

$$F = \text{All } x$$

$$n = 2$$

$$s(f,A) = A_F^x$$

$$R_F(a,b,c) = (a \cap b \subseteq c)$$

as a description of *all* in terms of Definition 2.

The pair of relations expressed in (46) and (48) together represent the set-theoretic relation expressed by *only*. A second "D" is not needed in (48), because its presence is already entailed by the first "D", intersected with $s(f,A)$, and the subset relation. We could express both (46) and (48) more compactly without mentioning D at all, but a good reason for including it in our set-theoretic relations will emerge from our discussion in IV,2.

Just as (43) is a special case of 11(100), the analysis in (44), similarly, is a special case of 11(130), obtained by taking $B=(x=y)$ and $A=F$. Formulas 11(130), however, constitute an analysis of *also*, not *even*, which Horn claims (44) is. Before investigating this discrepancy we can first notice that 11(130) is expressible in set theory, because its presupposition and assertion are both existential quantifications. Its assertion is (45), the presupposition of *only*, and is thus expressible in set theory as (46). Its presupposition is

$$(49) \quad (\text{Some } x)(-B;A)$$

which can be expressed set-theoretically as

$$(50) \quad D \cap (\text{Comp } B_F^x) \cap A_F^x \neq \lambda$$

because of the equivalence of (38) and (39). It follows that *also* is expressible in set theory.

Horn claims that *even* is analyzable as in (44), which we have just seen is expressible in set theory, but it is clear intuitively that *even* is not expressible in set theory. *Even* does presuppose (44P) and assert (44A), as Horn suggests, but the two conditions in (44) do not exhaust the meaning of *even*. *Even a is F* does presuppose that someone other than *a* is *F* and assert that *a* is *F*, as (44) says it does, but it also presupposes that *a*'s being *F* was unexpected. *Also a is F*, in contrast, has the same presupposition and assertion, exactly, that (44) gives to *even*, but it lacks this presupposition of unexpectedness. It follows that (44) is, in fact, an analysis of *also*, rather than *even*, whose semantic analysis requires both (44) and some analysis of the "psychological" or pragmatic notion of unexpectedness. It is precisely this "extra" element of meaning that makes us feel intuitively that *even* is not a quantifier. *Even a is F* not only gives an answer to the question *How many things (or people) have F?*, but it also tells us that the answer it does give is unexpected. *Also a is F* lacks this extra information and qualifies as a quantifier.

In exact accordance with these intuitions, the pragmatic notion unexpected cannot be expressed formally in set theory. *Even* expresses both the set-theoretic relations contained in (46) and (50) and the non-set-theoretic relation unexpected by the speaker. It follows that the quantifier *also* and the non-quantifier *even* differ only in that the former is set-theoretically expressible, while the latter is not.

Horn himself points out that "the difference between *also* and *even* is, of course, the notion of expectation presupposed by the latter" (p. 106), but he proposes an analysis of *even* that conspicuously ignores this fact, he gives no analysis of *also* at all, and he does not discuss the significance of the difference he points out. His failure to examine *also* and its relation to *even* in more detail leads him to make the erroneous claim that

the combination of the presupposition in (54) [our (44)] with the assertion $\neg Fx$ has no surface realization as such, but must emerge as something like

(55) *No, (others did but...) Muriel didn't.*

(p. 105)

As we saw in 11,3,3, the combination of the presupposition of *also* with the negation of its assertion yields the semantic analysis of the quantifier *only some...other than* and so does, in fact, have a surface realization.

CHAPTER 3: THE FORMAL DEFINITION OF QUANTIFIER

Section 1: Elementary Quantifiers

In the last two chapters we examined the two basic semantic characteristics of quantifiers, the binding property and their expressability in set theory. The binding property singles out one of the arguments of the propositional functions the quantifier operates on and set-theoretic expressability guarantees that what the quantifier says about this argument has something to do with the number of individuals that can serve successfully as its values. The truth-functional connectives, for example, are not quantifiers, because they are not binding. No matter how many arguments the members of an *n*-tuple of propositional functions may have, their conjunction and disjunction will each have the same number of arguments as the conjunct or disjunct with the largest number of arguments. There is no number such that the conjunction or disjunction of propositional functions with more than that number of arguments always has one fewer than that number of arguments. An operator like *even*, on the other hand, is binding, but it is still not a quantifier, as we saw in Chapter 2, because what it tells us about the bound argument has to do with more than the number of values that that argument can have. In other words, its meaning is not expressible in set theory.

We now have the basis for a formal explication of what it means to be a quantifier: a quantifier is a set-theoretically expressible binding operator. There is still one thing missing from our analysis, however, as we can see by comparing the semantic representations of quantificational sentences with their surface forms.

Each of the following sentences contains two or more instances of the quantifiers *all* or *some*:

- (51) *Some theorems can be proven by all logicians.*
- (52) *All theorems can be proven by some logicians in some systems.*
- (53) *Some theorems can be proven by some linguists in some systems.*

Their semantic representations, respectively, are

- (54) $(\text{Some } x)(\text{Some } y)(\text{Theorem}(x), \text{Logician}(y); \text{Provable}(x, y))$

(55) $(\forall x)(\exists y)(\forall z)(\text{Theorem}(x), \text{Logician}(y), \text{System}(z); \text{Provable}(x, y, z))$

(56) $(\exists x)(\exists y)(\exists z)(\text{Theorem}(x), \text{Linguist}(y), \text{System}(z); \text{Provable}(x, y, z))$

where "Provable" in (54) denotes the two-place relation *provable by* and in (55) and (56) the three-place relation *provable by in*. Each of the semantic representations has the same number of instances of "All" or "Some", respectively, as the sentence it represents has instances of *all* or *some*.

When we look more closely, however, an important difference emerges. There is no reason to consider any instance of *some* in (51), (52), or (53) to be any different from any other instance. Two of them make reference to theorems, two to logicians, one to linguists, and one to systems, but what they say about these different things is the same. Similarly, one instance of *all* in (52) makes reference to theorems and the other to systems, but they both seem clearly to be instances of the same semantic operator *all*.

The situation in (54), (55), and (56), however, is very different. According to Definitions 1 and 2, binding and set-theoretic expressability are properties of mappings, that is, functions, from R^n into R , a fact that is reflected in the form of our semantic representations. Viewed as mappings, however, neither all instances of "Some" in (54), (55), and (56) nor the two instances of "All" in (55) are the same.

Both instances of "Some" in (54) represent mappings from R^3 into R , but the first tells us something about the first argument, "x", of

(57) $(\text{Theorem}(x), \text{Logician}(x), \text{Provable}(x)),$

while the second tells us something about the second argument, "y". Both instances map the two-argument ordered triple (57) of propositional functions onto one-argument propositional functions, but the first maps it onto

(58) $(\exists x)(\text{Theorem}(x), \text{Logician}(y); \text{Provable}(x, y)),$

while the second maps it onto

(59) $(\text{Some } y)(\text{Theorem}(x), \text{Logician}(y); \text{Provable}(x, y)).$

Formulas (58) and (59) represent two very different one-argument propositional functions. Formula (58) represents the property of being a logician who is able to prove some theorem and it

does not even appear explicitly in (54), because the y-instance of "Some" applies to (57) first. Formula (59) represents the property of being a theorem that can be proven by some logician, a very different property from the one represented by (58). Since "Some x" and "Some y" provide different members of R for the same member of R^3 , they must be different mappings.

Similar remarks hold for the other quantifiers in (54), (55), and (56). "Some y" in (54) and (55) is manifested in both (51) and (52), respectively, as *some logicians*, but in (55) it is a mapping from R^4 into R , rather than a mapping from R^3 into R , as it is in (54). "All x" and "All z" in (55) are both mappings from R^4 into R , but they bind different arguments and so are different mappings. The three instances of "Some" in (56) are all mappings from R^4 into R , but they too are different mappings for the same reason.

Exactly the same situation holds for mappings from R into R . "Some x" maps the three-place relation

(60) $\text{Provable}(x, y, z),$

for example, onto the two-place relation

(61) $(\text{Some } x) \text{Provable}(x, y, z)$

while "Some y" and "Some z" map (60), respectively, onto

(62) $(\text{Some } y) \text{Provable}(x, y, z)$

(63) $(\text{Some } z) \text{Provable}(x, y, z).$

If we let

x = the Barcan formula

y = A. N. Prior

z = quantificational S_5

for example, for the free occurrences of "x", "y", and "z" in (60), (61), (62), and (63), then these formulas semantically represent, respectively, the sentences

(64) *The Barcan formula can be proven by Prior in quantificational S_5 .*

(65) *Something can be proven by Prior in Quantificational S_5 .*

- (66) *The Barcan formula can be proven by someone in quantificational S_5 .*
- (67) *The Barcan formula can be proven by Prior in something.*

These sentences clearly have very different meanings and (67) is even a little peculiar because of our failure, for the sake of simplicity, to specify that "z" ranges over the set of systems, as we did specify in (55) and (56). Since (61), (62), and (63) produce the very different sentences (65), (66), and (67), respectively, for the same assignment of individuals as values to variables, they must represent different relations or propositional functions. Since these different propositional functions are produced by quantification from the same propositional function (60), the quantifiers involved must be different. We see that something is clearly missing from our proposed formal characterization of quantifiers as set-theoretically expressible binding operators. Although *all* is simply *all* and *some* is just *some*, when they occur in English sentences, our formalism forces us to represent different occurrences of *all* and different occurrences of *some* by very different mappings.

This kind of situation might be considered acceptable, if we were concerned only with assigning semantic representations to sentences, as long as it was clear how to assign the appropriate mapping to a given occurrence of one of the quantifiers. All we would have to do would be to pick a different variable for each occurrence and keep our variables straight.

If we are concerned with answering the deeper theoretical question "What is a quantifier?", however, this situation poses a serious problem. Quantifiers in surface representation, that is, quantifiers as they actually appear in natural language, do not involve variables at all. Variables do not appear in sentences. In semantic representations of sentences, however, there can be "quantifiers" that differ only in that they occur with different variables. There is only one *some* in English and there is only one *all*, but our theory requires us to distinguish infinitely many "versions" of *some* and *all* by associating each quantifier with a variable every time it occurs.

A fully adequate answer to the question "What is a quantifier?" must tell us what it is about all different versions of *some* that makes them all "versions" of the same quantifier. We know that "All x", "All y", and "All z" are all instances of the same quantifier *all* and that each of these instances is a mapping from R^n into R , but we still have not determined what it is that makes these three mappings instances of the same quantifier, while the three mappings "All x", "Some y", and "Many z",

for example, are not. To answer the question "What is a quantifier?" we have to figure out what it is that makes some binding set-theoretically expressible mappings from R^n into R instances of the same quantifier and other such mappings instances of different quantifiers. We have to figure out how to "factor out" the variables to get at the quantifiers.

We can begin by making the following definition:

Definition 3 (Elementary Quantifier): Let F be a mapping from R^n into R . F is said to be an elementary quantifier in L if F is binding and expressible in set theory.

An elementary quantifier is the kind of thing we have been using to represent quantifiers in our semantic representations. Our problem now is to determine the conditions under which two elementary quantifiers represent in semantic representation the same intuitive quantifier in surface representation.

The solution that immediately suggests itself is to say that two elementary quantifiers represent the same quantifier if they both express the same set-theoretic relation. Since elementary quantifiers are set-theoretically expressible, there is a set-theoretic relation expressed by each one and it is the set-theoretic relation that expresses, in some sense, the meaning of the quantifier. Since elementary quantifiers represent the same surface quantifier if they differ only in the variable that they bind, it makes sense to take the precise notion of set-theoretic relation identity as the criterion of surface quantifier identity. This will, in fact, turn out to be the solution to our problem, but there are a number of technical intricacies that must be untangled, before we can make the solution precise formally.

Section 2: Normal Forms

We cannot say simply that two elementary quantifiers represent the same quantifier if they express the same set-theoretic relation, because the number of propositional functions that elementary quantifiers operate on can differ, even if the elementary quantifiers bind the same variable. The sentences

- (68) *All theorems can be proven.*
- (69) *All theorems that have been proven have been published.*

for example, have the respective semantic representations

(70) $(\forall x)(\text{Theorem}(x); \text{Provable}(x))$

(71) $(\forall x)(\text{Theorem}(x), \text{Proven}(x); \text{Published}(x))$

The elementary quantifier "All x" occurs in both semantic representations and clearly represents the same surface quantifier *all* in both cases. Despite this, however, "All x" in (70) expresses a different set-theoretic relation from that expressed by "All x" in (71).

The set-theoretic relation expressed by "All x" in (70) is given by

$$(72) \quad R_{\forall x}(a, b, c) = (a \cap b \subseteq c)$$

under the set assignment

$$(73) \quad s(f, A) = \frac{A_f^x}{\langle DR \rangle}$$

as we saw in 111,2,3. Since (70) is a mapping from R^2 into R , it expresses a set-theoretic relation that involves three sets. The same reasoning that gave us (72) also shows that the set-theoretic relation expressed by "All x" in (71) is given by

$$(74) \quad R_{\forall x}(a, b, c, d) = (a \cap b \cap c \subseteq d)$$

under the same set assignment (73). Since "All x" in (71) maps R^3 into R , it expresses a set-theoretic relation that involves four sets. Since the set-theoretic relations (72) and (74) involve different numbers of sets, they must be different relations. The two elementary quantifiers "All x" express different set-theoretic relations in (70) and (71), despite the fact that they represent the same surface quantifier *all*.

The relations in (72) and (74), though different, are also clearly very similar. They involve different numbers of sets, but what they say about those sets is the same: the intersection of all but the last of them is a subset of that one. If we can figure out a way to express this kind of similarity formally, then we may be able to save the notion that similarity of set-theoretic relations is the criterion for quantifier identity.

As we saw in 11,4, the main effect of an elementary quantifier is to express a relation between the different classes of propositional functions that are separated by semi-colons. The propositional functions that are separated by commas always end up being conjoined in the semantic analysis, regardless of which quantifier is involved.

Formula (70), for example, is equivalent to the formula

(75) $(\forall x)(\text{Theorem}(x) \supset \text{Provable}(x))$

in which the semi-colon has been replaced by material implication. Formula (71), similarly, is equivalent to the formula

(76) $(\forall x)((\text{Theorem}(x) \wedge \text{Provable}(x)) \supset \text{Published}(x))$

in which the semi-colon has again been replaced by material implication and the comma has been replaced by conjunction. If we leave the semi-colons in and only change the commas to conjunctions, then (70) remains the same, but (71) becomes

(77) $(\forall x)(\text{Theorem}(x) \wedge \text{Provable}(x); \text{Published}(x))$.

The form (77) works, in fact, even for those quantifiers, such as *many* and *most*, that are not reducible, as is easily verified. We cannot eliminate the semi-colon in such cases, as we can with *all*, but we can always turn the commas into conjunctions.

Formula (77) contains exactly the same number of quantified propositional functions or wffs as does (70). It also expresses the same set-theoretic relation under the same set assignment, because the relation

$$(78) \quad \frac{(\phi \wedge \psi)_f^\alpha}{DR} = \frac{\phi_f^\alpha \wedge \psi_f^\alpha}{\langle DR \rangle \quad \langle DR \rangle}$$

always holds. We will see later that properties like (78) make (73) a particularly useful set assignment.

We can generalize these results to get a normal form for the representation of elementary quantifiers. Since the commas in a semantic representation act like conjunction and end up as intersection operators in the set-theoretic relations, we can guarantee uniform set-theoretic relations for elementary quantifiers that represent the same surface quantifier by requiring that the commas not appear in the normal form. This gives us the following definition:

Definition 4 (Normal Form): Let F and Q be elementary quantifiers such that F maps R^n into R , Q maps R^k into R , and there is a set assignment s such that both F and Q are expressible under s . Let R_F and R_Q be set-theoretic relations such that F is expressible as R_F under s and Q is expressible as R_Q under s . Q is said to be the normal form of F in L , written

(79) $Q = F_{\text{norm}}$ in L ,

if there are integers n_j , $j=0, \dots, k$, such that

$$0 = n_0 \leq n_1 \leq \dots \leq n_k = n$$

and for all $a_i \in 2^D$, $i=0, \dots, n$,

$$(80) \quad R_F(a_0, \dots, a_n) = R_Q(b_0, \dots, b_k),$$

where

$$b_0 = a_0, \quad b_j = \bigwedge_{i=n_{j-1}+1}^{n_j} a_i, \quad j=1, \dots, k,$$

and k is the least value for any such Q . If k is the minimal value such that (79) and (80) hold, then k is said to be the order of F in L and we write

$$k = \text{order}(F) \text{ in } L.$$

The normal form of an elementary quantifier, in other words, is the elementary quantifier which is equivalent to the given elementary quantifier, but whose semantic representation and analysis contain only semi-colons, all the commas having been rewritten as conjunctions.

The most general form of the universal (elementary) quantifier, for example, is, as we saw in 11.4.1.1,

$$(81) \quad (\text{All } x)(B_1, \dots, B_n; A_1, \dots, A_m)$$

the semantic analysis of which we gave as 11(171). The normal form of this elementary quantifier, however, according to Definition 4, is

$$(82) \quad (\text{All } x)(B; A).$$

Formula (82) has no commas, at least as far as the "All x " is concerned, and every instance of (81) can be rewritten as an instance of (82) by taking B to be the conjunction of the B_i 's and A the conjunction of the A_j 's.

More explicitly in terms of Definition 4 we first rewrite (81) as

$$(83) \quad (\text{All } x)(A_1, \dots, A_{n_1}; A_{n_1+1}, \dots, A_n)$$

in order not to confuse the n in (81) with the n in (80). In other words, we rename the n in (81) as the n_1 in (83) and we rename the m in (81) as the n in (83). We also rename the B 's as A 's for notational uniformity and we continue numbering what were A 's in (81) consecutively after the new A 's of (83). This makes the n in (83) equal to $n+m$ for the n and m of (81). In terms of the actual schemas (81) and (83), we are renaming B_1, B_n, A_1, A_m in (81) as $A_1, A_{n_1}, A_{n_1+1}, A_n$, respectively, in (83).

Again for the sake of uniformity, we will rewrite (82) in the form

$$(84) \quad (\text{All } x)(B_1; B_2)$$

replacing the B of (82) with B_1 and the A of (82) with B_2 .

Since (83) and (84) are just notational variants, respectively, of (81) and (82), anything we say about the set-theoretic relations of (83) and (84) will hold equally well for those of (81) and (82).

"All x " in (83) is a mapping from R^n into R and "All x " in (84) is a mapping from R^2 into R . This gives us $k=2$ in Definition 4. We know that (84) is expressible as (72) under the set assignment (73) and we can rewrite (72) as

$$(85) \quad R_{\text{All } x}(b_0, b_1, b_2) = (b_0 \cap b_1 \subseteq b_2)$$

again to get it into the notation of Definition 4. As we would expect, (85) confirms the fact that $k=2$.

By reasoning similar to that which led us to see that (70) is expressible as (72) under (73) and that (71) is expressible as (74) under (73), we can also see, inductively, that (83) is expressible as

$$(86) \quad R_{\text{All } x}(a_0, \dots, a_n) = (a_0 \cap \dots \cap a_{n_1} \subseteq a_{n_1+1} \cap \dots \cap a_n)$$

under (73). This, however, is exactly what Definition 4 requires of an elementary quantifier and its normal form.

First, we see that there are integers n_j , $j=0, 1, 2$, such that

$$0 = n_0 \leq n_1 \leq n_2 = n.$$

All we have to do is to take n_1 as in (83). If we now take

$$b_0 = a_0$$

$$b_1 = a_1 \wedge \dots \wedge a_{n_1}$$

$$b_2 = a_{n_1+1} \wedge \dots \wedge a_n$$

and if we rename the $R_{All\ x}$ of (85) as $R_{All\ x}^2$ and the $R_{All\ x}$ of (86) as $R_{All\ x}^1$ just to keep them separate, then we get the relation

$$(a_0 \wedge \dots \wedge a_{n_1} \subseteq a_{n_1+1} \wedge \dots \wedge a_n) = (b_0 \wedge b_1 \subseteq b_2)$$

and its more compact form

$$(87) \quad R_{All\ x}^1(a_0, \dots, a_n) = R_{All\ x}^2(b_0, b_1, b_2).$$

Again we see that the elementary quantifier represented by (82) or (84) is the normal form of the elementary quantifier represented by (81) or (83).

It follows from (87) that the order of "All x " is 2. Similar reasoning reveals that the order of "Some x " is 1, because even the semi-colon turns into conjunction in that case, and the order of "The x " is 3, because its semantic analysis requires, as we saw in 14(181), three differently functioning classes of wffs. It is also not difficult to see that the only elementary quantifiers expressible under the set assignment (73) that have English surface forms and are of order 1 are the numerical quantifiers and those of the form "Some α " or "No α ".

The integers n_j in Definition 4 mark off the differently functioning classes of wffs or propositional functions in a semantic representation. If the wffs in a semantic representation are numbered consecutively beginning with 1, then the integer n_j is the subscript of the wff that appears immediately to the left of the j^{th} semi-colon (or the right-most parenthesis, if $n_j = n$). We have succeeded in abstracting away from the number of wffs in a semantic representation, that is, the number of propositional functions operated on by an elementary quantifier, so that only the set-theoretic relation remains.

Section 3: The Standard Set Assignment

Now that we have a normal form for the representation of elementary quantifiers, we can represent every elementary quantifier by its normal form. We can try to formulate our criterion of quantifier identity in terms of the identity of set-theoretic relations expressed by the normal forms of elementary quantifiers, identifying elementary quantifiers that differ only in the number of propositional functions they operate on. There is still one further obstacle, however, that has to be taken care of. Set-theoretic expressability is formulated in Definition 2 in its most general form, in terms of both set-theoretic relations and set assignments. The problem now is that the same set-theoretic relation can be expressed by very different elementary quantifiers under different set assignments. Just as we normalized the number of wffs an elementary quantifier operates on, we also have to standardize our set assignments in some way.

As an example, we can take the set-theoretic relation

$$(88) \quad R_c(a, b, c) = (a \wedge b \subseteq c)$$

which we saw in the last section as (72) and the notational variant (85). Formula (88) gives us the set-theoretic relation expressed by normal *all* under the set assignment (73), reproduced here as

$$(89) \quad s_1(f, A) = \frac{A_f^\alpha}{\langle DR \rangle}$$

R_c , in other words, is the set-theoretic relation expressed by the elementary quantifier F_1 that maps (A_1, A_2) onto $(All\ \alpha)(A_1; A_2)$.

Under other set assignments, however, (88) is expressed by other elementary quantifiers. If, instead of (89), we take the set assignment

$$(90) \quad s_2(f, A) = \text{Comp} \frac{A_f^\alpha}{\langle DR \rangle}$$

which is perfectly consistent with Definition 2, then (88) is expressed by the different elementary quantifier F_2 that maps (A_1, A_2) onto $(All\ \alpha)(A_2; A_1)$. Both F_1 and F_2 are set-theoretically expressible binding mappings from R^2 into R , so both of them are elementary quantifiers. Both express the same set-theoretic relation (88), but they are different elementary quantifiers, because they produce different results from the same pair of wffs. They manage to express the same set-theoretic relation, because they do so under different set assignments.

More complicated examples can also be constructed. If we take the set assignment defined by

$$(91) \quad s_3(f, A) = \begin{cases} A_f^\alpha & \text{if } m \text{ is even} \\ \langle DR \rangle & \\ \text{Comp } A_f^\alpha & \text{if } m \text{ is odd} \\ \langle DR \rangle & \end{cases}$$

where m is the number of arguments in A , as in Definition 1, then we get the elementary quantifier defined by

$$(92) \quad F_3(A_1, A_2) = \begin{cases} F_1(A_1; A_2) & \text{if } m_1, m_2 \text{ are both even} \\ (All \alpha)(A_1; -A_2) & \text{if } m_1 \text{ is even, } m_2 \text{ is odd} \\ (All \alpha)(-A_1; A_2) & \text{if } m_1 \text{ is odd, } m_2 \text{ is even} \\ F_2(A_1; A_2) & \text{if } m_1, m_2 \text{ are both odd.} \end{cases}$$

Like both F_1 and F_2 , F_3 is an elementary quantifier of order 2. Also like F_1 and F_2 , F_3 expresses the same set-theoretic relation R defined in (88), but under the different set assignment (91).

By choosing the appropriate set assignment, we can even find non-binding operators that express R_c . This constitutes a proof of the logical independence of the binding and set-theoretic expressability properties and justifies our decision to express Definition 2 in its full generality. Under the set assignment

$$(93) \quad s_4(f, A) = \begin{cases} D & \text{if } f \text{ satisfies } A, \\ \lambda & \text{if } f \text{ satisfies } -A \end{cases}$$

for example, we get the following instances of (88):

$$(94) \quad D \cap D \subseteq D \text{ if } f \text{ satisfies } A_1 \text{ and } f \text{ satisfies } A_2$$

$$(95) \quad D \cap D \subseteq \lambda \text{ if } f \text{ satisfies } A_1 \text{ and } f \text{ satisfies } -A_2$$

$$(96) \quad D \cap \lambda \subseteq D \text{ if } f \text{ satisfies } -A_1 \text{ and } f \text{ satisfies } A_2$$

$$(97) \quad D \cap \lambda \subseteq \lambda \text{ if } f \text{ satisfies } -A_1 \text{ and } f \text{ satisfies } -A_2$$

Of these formulas, however, (94), (96), and (97) are true, no matter what A_1 and A_2 are, as long as the respective condition on f holds. Formula (45), in contrast, is always false, independently of A_1 and A_2 , as long as f satisfies the first but not the second. It follows that R is expressed under s_4 by the mapping defined by

$$(98) \quad F_4(A_1; A_2) = \begin{cases} \text{falsity under } f \text{ if } f \text{ satisfies } A_1 \\ \text{but not } A_2 \\ \text{truth under } f \text{ otherwise.} \end{cases}$$

This, however, is the definition of material implication. In other words " $F_4(A_1; A_2)$ " is just another way of writing " $A_1 \supset A_2$ ". This mapping is not an elementary quantifier, because it is not binding, but it expresses R_c under s_4 all the same.

As a final example, we can get much more specific and take D to be the set of non-negative integers. If we take the set assignment

$$(99) \quad s_5(f, A) = \text{the set of integers which have the same parity as } m$$

then R_c is expressed under s_5 by the mapping defined by

$$(100) \quad F_5(A_1; A_2) = \begin{cases} \text{truth if } m_1, m_2 \text{ have the same parity} \\ \text{falsity otherwise.} \end{cases}$$

If A_1 and A_2 both have an even number of arguments or both have an odd number of arguments, then $F_5(A_1; A_2)$ is logically true.

Otherwise, it is logically false. R_c turns out to be just a variant of the relation "have the same parity" that can hold between integers. Like F_4 , F_5 is not an elementary quantifier, because it is not binding.

We began with a single set-theoretic relation (88), and we have managed to find five different set-theoretically expressible mappings that express that relation. Three of these are binding and so are elementary quantifiers and two are not binding and so are not elementary quantifiers. Clearly an infinite number of such mappings, both binding and non-binding, can be constructed, so we cannot take identity of set-theoretic relations as our criterion of quantifier identity, even if we restrict ourselves to normal forms.

If we examine our five mappings carefully, however, we see that the three binding operators arise from set assignments that are defined, directly or indirectly, in terms of the set assignment (89), while the two non-binding operators are defined independently of (89). The joint action of binding and set-theoretic expressability produces operators that bind an argument and say something about the size of its extension. In other words, the correlation between binding and (89) is no accident.

The set assignment (89) maps each propositional function onto its extension with respect to the bound argument or variable. Any other binding mapping, if it binds the same argument, must also make reference to that extension. This is exactly the semantic significance of the binding property.

Because of this basic character of (89) or (73), we introduce the following definition:

Definition 5 (Standard Set Assignment): Let s^* be the set assignment defined by

$$(101) \quad s^*(f, \phi) = \phi_f^\alpha \cdot \langle DR \rangle$$

s^* is said to be the standard set assignment.

The standard set assignment enables us to correlate elementary quantifiers with their set-theoretic relations in an intuitively natural way. As is clear from formula (101), it is really a schema depending on α . In other words, there are infinitely many standard set assignments, one corresponding to each variable or argument that could be quantified, but what (101) says about each such argument is the same. Other set assignments are definitely of interest from a mathematical and meta-theoretical point of view, in our proof of the independence of set-theoretic expressability and binding, for example, but (101) is all we need to complete our formal characterization of quantifiers.

Section 4: Quantifiers as Equivalence Classes

In Section 1 we defined an elementary quantifier as a set-theoretically expressable binding operator and we pointed out that a surface quantifier seems intuitively to be associated in some way with the set-theoretic relation that is expressed by the elementary quantifier that appears in its place in semantic representation. In Section 2 we noted that there can be elementary quantifiers that seem intuitively to express the same set-theoretic relation, but that differ in the number of propositional functions they operate on and so are different elementary quantifiers. We got around this problem by defining a normal form for elementary quantifiers through abstraction away from the number of propositional functions. What we did, in essence, was to partition the class of elementary quantifiers into equivalence classes with equivalence taken to be the property of having the same normal form. A normal form was then taken to be representative of all elementary quantifiers whose normal form it is.

In Section 3 we noted the further problem that a given set-theoretic relation can be expressed by any number of different elementary quantifiers of different normal forms by choosing

different set assignments. We got around this problem by choosing s^* as the standard set assignment because of its intuitive interpretation as the extension of a propositional function or wff with respect to a given argument or variable and because of its occurrence as a part of the other set assignments that produce binding mappings. This solution can also be interpreted in terms of equivalence classes. If we say that two elementary quantifiers are equivalent if they express the same set-theoretic relation under some set assignment, then each set assignment defines an equivalence class in a partition of the class of elementary quantifiers. In this case, however, we singled out one of these equivalence classes for special attention, rather than singling out one member of each class as representative of its class.

The formal definition of quantifier can be obtained by taking yet another equivalence-class partition of the class of elementary quantifiers. What we have really done in the last two sections is to determine how to express in formal terms the intuitive notion that elementary quantifiers represent the same surface quantifier if they express the same set-theoretic relation, that is, that quantifiers are determined by set-theoretic relations. This is formulated explicitly in the following definitions:

Definition 6 (Relational Equivalence): Let q_1 and q_2 be elementary quantifiers and let s be a set assignment such that both q_1 and q_2 are expressable in set theory under s . We say that q_1 and q_2 are relationally equivalent under s if the same set-theoretic relation is expressed under s by their normal forms.

Definition 7 (Standard Elementary Quantifier): Let q be an elementary quantifier. We say that q is standard if it is expressable in set theory under s^* .

Definition 8 (Quantifier): Let Q be a set of standard elementary quantifiers. Q is said to be a quantifier if there is a standard elementary quantifier q such that every member of Q is relationally equivalent to q under s^* and Q is closed under relational equivalence under s^* . The order of Q , written

$\text{order}(Q)$,

is defined to be the order of any of its members.

Definition 9 (Quantificational Relation): Let Q be a quantifier and let q be a member of Q . The set-theoretic relation expressed by the normal form of q under s^* is said to be the quantificational relation expressed by Q .

In accordance with our previous discussion, Definition 6 says that we will treat two elementary quantifiers as the same entity under s if their normal forms express the same set-theoretic relation under s . This means that we will really be talking about normal forms in what follows. Definition 7 fixes s as s^* , because this is the only s that really corresponds to our intuitions about quantifiers, and it tells us to be concerned only with those elementary quantifiers that are expressible under s^* . Definition 8 identifies a quantifier with the set-theoretic relation that is expressed by the elementary quantifiers that represent it in semantic representation, subject to the foregoing comments about normal forms and the standard set assignment. Definition 9 underscores this identity by explicitly associating the quantifier with the relation.

From one point of view, Definition 8 is the culmination of our study, because it provides us with a clear and precise explication of what a quantifier is. In fact, however, a lot remains to be done. As we will see in Part IV, the main significance of Definition 8 is really that it provided the background that makes Definition 9 possible. Quantificational relations provide us, as we will see, with the basis for an explanatory formal theory of quantifiers in natural language, including a principled way to represent quantifiers and modal adverbs in the lexicon of a generative grammar.

Part IV

The Theory of Quantificational Relations

CHAPTER 1: LEXICAL REPRESENTATION OF QUANTIFIERS

Section 1: Explicit Forms of Quantificational Relations

The discussion in the last chapter was rather abstruse, but it was important, because it provided us with a rigorous theoretical foundation for the key notion of quantificational relation. The quantificational relation expressed by a quantifier is the relation that must hold among the extensions of the wffs or propositional functions operated on by the quantifier in order for the quantification in which the quantifier occurs to be true. In the case of multi-untruth-valued quantifiers, this notion can be generalized in the obvious way. More important than this general correlation between quantifiers and their quantificational relations, however, is the specific fact that quantificational relations contain all of the semantic information that needs to be included in the lexical entries of quantifiers.

As an example, let us consider the quantificational relation expressed by the universal quantifier. We saw in III,3,2 that *all* is of order 2 and that the quantificational relation that is expressed by it is given by

$$(1) R_{all}(b_0, b_1, b_2) = (b_0 \cap b_1 \subseteq b_2)$$

under s^* . We can express R_{all} in an explicit form, without reference to the set arguments b_i , by performing some formal manipulations to "factor out" the component relations that make it up. In other words, we can express R_{all} in algorithmic form, as a sequence of consecutive operations applied to the set triple (b_0, b_1, b_2) . Clearly this will do away with the artificial notational difference between formulas like III(72) and III(85).

To begin with, we can "factor out" the basic relation of subset,

$$\subseteq$$

to get

$$(2) R_{all}(b_0, b_1, b_2) = \subseteq(b_0 \cap b_1, b_2)$$

as a reformulation of (1). This says that R_{all} is the relation of subset, holding between the two sets $b_0 \cap b_1$ and b_2 . Next we can "factor out" the intersection relation, specifying that it holds of the first two sets in the triple (b_0, b_1, b_2) . This can be indicated by rewriting (2) in the form

$$(3) R_{all}(b_0, b_1, b_2) = (\subseteq, n_{12})(b_0, b_1, b_2, b_3)$$

where the subscripts "12" tell us that the intersection applies to the 1st and 2nd set arguments of the triple. What (3) tells us is that the effect of R_{all} on the triple of sets (b_0, b_1, b_2) is achieved by first taking the intersection of b_0 and b_1 and then taking that intersection to be a subset of b_2 .

The usefulness of (3) consists in the fact that the set arguments appear explicitly on both sides of the equation. As a result, we can omit them altogether to get the more concise form

$$(4) R_{all} = (\subseteq, n_{12})$$

The only information that (3) contains that (4) lacks is the order of the quantifier, that is, the number of wffs that the quantifier's normal form operates on. This is one less than the number of set arguments that the quantificational relation operates on, as we have seen. We can correct this defect in (4) by simply adding an initial order component to get

$$(5) R_{all} = (2; \subseteq, n_{12})$$

as the complete specification of R_{all} . The semi-colon in (5) serves merely to keep the order component separate from the rest of the relation and is related to the semi-colons we have used before only indirectly, through the definition of order. We will also find it useful to use the same algorithmic factoring procedure to get explicit forms of the set-theoretic relations expressed by elementary quantifiers that are not normal forms. To be consistent with the use of an order component in quantificational relations, we will use an initial integer in these other cases to indicate one less than the number of set arguments taken by the relation.

Explicit forms can be obtained for each of the other quantificational relations we have seen in a similar way. In III,2,3 we saw that the set-theoretic relation expressed by

$$(6) (\text{Some } x)(B;A)$$

under s^* is given by

$$(7) R_{\text{some } x}(b_0, b_1, b_2) = (b_0 \cap b_1 \cap b_2 \neq \lambda)$$

In III,3,2 we noted that *some* is of order 1, according to Definition 4, so we can get R_{some} , the quantificational relation expressed by *some*, from the normal form

$$(8) (\text{Some } x) B$$

rather than from (6). The set-theoretic relation expressed by (8) under s^* is given by

$$(9) R_{\text{some}}(b_0, b_1) = (b_0 \cap b_1 \neq \lambda)$$

but if we keep in mind that b_0 is always interpreted as D, then we can reformulate (9) as

$$(10) R_{\text{some}}(b_0, b_1) = (b_0 \cap b_1 \neq \text{Comp } b_0)$$

since $\lambda = \text{Comp } D$. The symbol "Comp" denotes the function that maps a set onto its complement, the set of individuals that do not belong to it.

Now we can factor out the relations in (10). First we can take out the \neq to get

$$R_{\text{some}}(b_0, b_1) = \neq(b_0 \cap b_1, \text{Comp } b_0)$$

and then we can reorder the arguments of \neq to get

$$R_{\text{some}}(b_0, b_1) = \neq(\text{Comp } b_0, b_0 \cap b_1)$$

The reordering is possible because $=$ is symmetric and we do it to get the two occurrences of b_0 together. This will simplify things a little later on.

Now we factor out first the Comp and then the \cap , using subscripts again to indicate which arguments they refer to at this stage. This gives us, in sequence,

$$R_{\text{some}}(b_0, b_1) = (\neq, \text{Comp}_1)(b_0, b_0 \cap b_1)$$

$$R_{\text{some}}(b_0, b_1) = (\neq, \text{Comp}_1, n_{23})(b_0, b_0, b_1)$$

Now the reason for the reordering becomes clear. We can collapse the two occurrences of b_0 into one by factoring out the function Pair_1 , which reduplicates the first member of an n -tuple. This gives us the explicit form

$$R_{\text{some}}(b_0, b_1) = (\neq, \text{Comp}_1, n_{23}, \text{Pair}_1)(b_0, b_1)$$

in which the ordered pair (b_0, b_1) appears explicitly on both sides of the equation. As a result we can drop it altogether and add the order of *some* as an initial component to get

$$(11) \quad R_{\text{some}} = (1; \neq, \text{Comp}_1, n_{23}, \text{Pair}_1)$$

as our complete specification of R_{some} .

In III,2,2 we saw that the set-theoretic relation expressed by simple *Most* α under s^* is given by

$$(12) \quad R_{\text{Most } \alpha}(b_0, b_1) = (K(b_0 \cap b_1) > K(b_0 \cap \text{Comp } b_1))$$

The normal form of *Most* α , however, is of order 2, because all instances of *most* involve two differently functioning classes of wffs, so we get

$$(13) \quad R_{\text{most}}(b_0, b_1, b_2) = (K(b_0 \cap b_1 \cap b_2) > K(b_0 \cap b_1 \cap \text{Comp } b_2))$$

in place of (12) as the quantificational relation expressed by *most*.

Now we begin the process of factorization. The first thing to take out of (13) is clearly $>$, giving us

$$R_{\text{most}}(b_0, b_1, b_2) = >(K(b_0 \cap b_1 \cap b_2), K(b_0 \cap b_1 \cap \text{Comp } b_2))$$

Next we take out the function K , which maps each member of an ordered n -tuple of sets onto its cardinality, and we get

$$R_{\text{most}}(b_0, b_1, b_2) = (>, K)(b_0 \cap b_1 \cap b_2, b_0 \cap b_1 \cap \text{Comp } b_2)$$

Now we take out the two intersection functions, which differ only in their subscripts, to get

$$(14) \quad R_{\text{most}}(b_0, b_1, b_2) = (>, K, n_{234}, n_{123})(b_0, b_1, b_2, b_0, b_1, \text{Comp } b_2)$$

The right-most intersection function operates first, to turn

$(b_0, b_1, b_2, b_0, b_1, \text{Comp } b_2)$ into $(b_0 \cap b_1 \cap b_2, b_0, b_1, \text{Comp } b_2)$ and then the other intersection function operates, to turn this 4-tuple into $(b_0 \cap b_1 \cap b_2, b_0 \cap b_1 \cap \text{Comp } b_2)$. Taking the function Comp_6 out of (14) now gives us

$$R_{\text{most}}(b_0, b_1, b_2) = (>, K, n_{234}, n_{123}, \text{Comp } 6)$$

$$(b_0, b_1, b_2, b_0, b_1, b_2)$$

and taking Pair_{123} , which reduplicates the first three members of an n -tuple, out of this formulation produces

$$(15) \quad R_{\text{most}}(b_0, b_1, b_2) = (>, K, n_{234}, n_{123}, \text{Comp } 6, \text{Pair}_{123})(b_0, b_1, b_2)$$

which has the argument triple (b_0, b_1, b_2) occurring explicitly on both sides of the equality. This again enables us to drop the set arguments entirely from (15) and, adding an initial order component, we get

$$(16) \quad R_{\text{most}} = (2; >, K, n_{234}, n_{123}, \text{Comp } 6, \text{Pair}_{123})$$

as the quantificational relation expressed by *most*.

Section 2: Manifold Quantifiers

Altham gives intuitive reasons for thinking "that a natural way to analyze sentences involving plural quantification is in terms of notions belonging to set theory" (pp. 7-8), but he immediately goes wrong, when he tries to formalize this idea for *many*. He introduces a new primitive symbol, "M", meaning "is a manifold", and transcribes a quantification of the form

$$(17) \quad \text{Many } B\text{'s are } A\text{'s}$$

in the form

$$(18) \quad (\exists x)(Mx \& (\forall y)(y \in x \rightarrow (B(y) \& A(y))))$$

The sentence

Many men are lovers.

for example, is written by Altham in the form

$$(\exists x)(Mx \& (\forall y)(y \in x \rightarrow y \text{ is a man and } y \text{ is a lover}))$$

Such a formulation is not strictly set-theoretic, however, because it depends on the new notion *M*, which, though definable in set theory, does not "belong" to set theory proper.

Even if we correct this deficiency in Altham's formulation, however, we still find ourselves on the wrong track. We can make (18) strictly set-theoretic by using "*M*" as a set variable, rather than as a primitive predicate symbol, and defining "manifold" explicitly in terms of this new "*M*". This gives us

$$(19) (\exists M) (M \subseteq D \wedge K(M) = n \wedge (\forall x) (x \in M \supset (B(x) \wedge A(x))))$$

instead of (18), as an analysis of *many* "in terms of notions belonging to set theory," as Altham would say. The number *n*, of course, is the manifold size index and *K* is again the cardinality function. If we try to derive an explicit set-theoretic relation from (19), however, we find that this is impossible, because of the quantificational character of (19) itself. The presence of quantifiers in the formula that gives us the set-theoretic relation, that is, (19), makes it impossible to factor out any of the component relations to get an explicit relation depending on *D*, *B*, and *A*.

We can still get an explicit set-theoretic relation expressed by *many*, however, by going back and starting over again. The problem with Altham's formulation (18), and our more precise formulation (19) is that they are unnecessarily complicated, using quantifiers to say something that could be said just as well without them (This fact was pointed out to me, in a different context, by David Kaplan (personal communication)). Our analysis of *many* in 11.2.3.2 says that (17) is true under an interpretation if and only if there are *n* distinct *t*-variants of that interpretation that satisfy *B*, every one of which satisfies *A*. Set-theoretically, all this means is that there must be at least *n* distinct individuals in the extension of *B* with respect to the quantified variable that are also in the extension of *A* with respect to that variable. In other words, the number of individuals that belong to both of these extensions must be at least *n*. It follows that the very simple formula

$$(20) K(D \cap B_f^\alpha \cap A_f^\alpha) \geq n$$

gives us the set-theoretic relation expressed by relativized *many* under *s**. Since *many*, like *most*, is clearly of order 2, we get (20) also as the quantificational relation expressed by *many*.

In contrast with the quantificational (19), the non-binding (20) is easily made explicit. First we factor out the relation "at least *n*" to get

$$(\geq n) (K(D \cap B_f^\alpha \cap A_f^\alpha))$$

and then we factor out the cardinality function to get

$$(\geq n, K) (D \cap B_f^\alpha \cap A_f^\alpha)$$

Finally, we take out the intersection relation to get

$$(\geq n, K, \cap_{123}) (D, B_f^\alpha, A_f^\alpha)$$

and we drop the explicit set arguments and add an initial order component to get

$$(21) R_{\text{many}} = (2; \geq n, K, \cap_{123})$$

as the quantificational relation expressed by *many*.

Section 3: Quantificational Relations and Lexical Entries

As we noted in Section 1, quantificational relations contain all of the semantic information that needs to be included in the lexical entries of quantifiers. If we assume that the results of our general logical analysis of quantifiers in 111 is included within universal semantic theory, then we can construct semantic analyses of specific quantifiers, of the traditional sort that we examined in 11, directly from the explicit quantificational relations that we derived in the last two sections. The quantifier specific information, which is what must be included in the lexical entry, is entirely contained in the quantificational relation.

The semantic part of the lexical entry for *all*, for example, would be the ordered triple (5) and the obvious conventions for transforming this ordered triple into a semantic analysis would be incorporated in universal semantic theory on the basis of our discussion in the last two Parts. We could then start with (5) and go through the following derivation:

$$(2; \subseteq, \cap_{12})$$

$$(\subseteq, \cap_{12}) (b_0, b_1, b_2)$$

$$(\subseteq) (b_0 \cap b_1, b_2)$$

$$b_0 \cap b_1 \subseteq b_2$$

$$D \cap \Psi_f^\alpha \subseteq \Phi_f^\alpha$$

$$\langle DR \rangle \langle DR \rangle$$

$$D \cap E_x[f_x^\alpha \text{ satisfies } \Psi \text{ in } \langle DR \rangle] \subseteq E_x[f_x^\alpha \text{ satisfies } \Phi \text{ in } \langle DR \rangle]$$

$$(22) E_x[x \in D \text{ and } f_x^\alpha \text{ satisfies } \Psi \text{ in } \langle DR \rangle] \subseteq E_x[f_x^\alpha \text{ satisfies } \Phi \text{ in } \langle DR \rangle].$$

Formula (22) gives us a Kaplan-style analysis of relativized *all*, which is, in fact, the normal form of *all*. The statement

f satisfies $All \alpha(\Psi; \Phi)$ if and only if (22)

is logically equivalent to each of the notationally variant analyses of relativized *all* that we saw in 11, 1, 3. We see that Kaplan's notational framework follows naturally from the theory of quantificational relations, as we have developed it in this study.

The derivation that we just saw was constructed by reversing the process by which we constructed the quantificational relation (5) in the first place. It illustrates how we can construct a similar derivation for any such explicit quantificational relation. *Most*, for example, is an even clearer case than *all*, because the semantic analysis of *most* that we have examined is already formulated in Kaplan's notation. We begin with the relation (16) and then unravel, step by step, the various formalizations that are encapsulated within it, arriving at the derivation

$$\begin{aligned}
 & (2; >, K, n_{234}, n_{123}, Comp_6, Pair_{123}) \\
 & (>, K, n_{234}, n_{123}, Comp_6, Pair_{123}) (b_0, b_1, b_2) \\
 & (>, K, n_{234}, n_{123}, Comp_6,) (b_0, b_1, b_2, b_0, b_1, b_2) \\
 & (>, K, n_{234}, n_{123}) (b_0, b_1, b_2, b_0, b_1, Comp b_2) \\
 & (>, K, n_{234}) (b_0 \cap b_1 \cap b_2, b_0, b_1, Comp b_2) \\
 & (>, K) (b_0 \cap b_1 \cap b_2, b_0 \cap b_1 \cap Comp b_2) \\
 & (>) (K(b_0 \cap b_1 \cap b_2), K(b_0 \cap b_1 \cap Comp b_2)) \\
 & K(b_0 \cap b_1 \cap b_2) > K(b_0 \cap b_1 \cap Comp b_2) \\
 & K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) > K(D \cap \Psi_f^\alpha \cap Comp \Phi_f^\alpha) \\
 & \quad \langle DR \rangle \quad \langle DR \rangle \quad \quad \langle DR \rangle \quad \quad \langle DR \rangle \\
 & K(D \cap \mathbb{E}_X^\alpha [f_X^\alpha \text{ satisfies } \Psi \text{ in } \langle DR \rangle] \cap \mathbb{E}_X^\alpha [f_X^\alpha \text{ satisfies } \Phi \\
 & \quad \text{in } \langle DR \rangle]) > K(D \cap \mathbb{E}_X^\alpha [f_X^\alpha \text{ satisfies } \Psi \text{ in } \langle DR \rangle] \\
 & \quad \cap \mathbb{E}_X^\alpha [f_X^\alpha \text{ satisfies } \neg \Phi \text{ in } \langle DR \rangle]) \\
 (23) \quad & K(\mathbb{E}_X^\alpha [x \in D \text{ and } f_X^\alpha \text{ satisfies } \Psi \text{ in } \langle DR \rangle \text{ and } f_X^\alpha \text{ satisfies } \Phi \\
 & \quad \text{in } \langle DR \rangle]) > K(\mathbb{E}_X^\alpha [x \in D \text{ and } f_X^\alpha \text{ satisfies } \Psi \text{ in } \langle DR \rangle \text{ and } \\
 & \quad f_X^\alpha \text{ satisfies } \neg \Phi \text{ in } \langle DR \rangle]).
 \end{aligned}$$

The last line (23) of this derivation is exactly the analysis of relativized *most* that we constructed in 11, 2, 3.1.

Given an explicit quantificational relation like (5), (11), (16), or (21), our theory automatically provides us with a complete semantic analysis of the corresponding quantifier. It follows that we can take these quantificational relations as the lexical entries of quantifiers in a generative grammar, insofar as these contain semantic information. Exactly which set-theoretic relations, such as cardinality, subset, greater than, and "pair", are required is an empirical question, whose answer is still to be determined. It is clear, however, that there will be a very limited number of such relations and these can be taken to be substantive universals in semantic theory.

CHAPTER 2: SIMPLIFICATION AND RELATIVIZATION

Section 1: Simple Quantification and the Universal Domain

Throughout this study we have equivocated on the meaning of simple quantification. Given a simple quantification like

(24) (Some x) A

we have sometimes interpreted the quantifier involved to mean *something* and we have sometimes interpreted it to mean *someone*, without ever specifying how to choose the appropriate interpretation in specific cases. The semantic representation of

(25) *Something is provable.*

we have taken to be

(26) (Some x) Provable(x)

and the semantic representation of

(27) *Someone is a linguist.*

we would take to be

(28) (Some x) Linguist(x)

but no principled basis has been provided for explaining why the "Some x" in (26) is realized as *something* in (25), while the "Some x" in (28) is realized as *someone* in (27). It turns out that the quantificational relations introduced in Definition 9 provide us with a very natural framework for expressing and explaining the distinction between simple and relativized quantifiers, on the one hand, and the distinction between the varieties of simple quantification, on the other.

Intuitively, it is clear that the difference between (26) and (28) lies in the predicate, that is, in the quantified wff. We interpret "Some x" in (28) as *someone*, because only people can be linguists and *someone* refers to people. We interpret "Some x" in (26) as *something*, because only non-people, that is, things, can be provable and *something* refers to things. The problem is how to express this very simple intuition in formal terms.

The problem is easily solved in terms of quantificational relations. A quantificational relation, formally speaking, is

an ordered set, whose first member is a positive integer, the order of the quantifier, and whose other members constitute an algorithmic sequence of consecutively applied set-theoretic operators. These operators operate on (n+1)-tuples of sets, the first of which must always be D, variously called by logicians the domain of discourse, and by mathematicians the universal set. We will call D the universal domain as a compromise.

The universal domain is the set of all individuals that can be talked about, that is, that can be referred to by a bound variable like "x". It follows that we can distinguish between the simple quantification in (26) and the simple quantification in (28) by allowing our theory to recognize two universal domains, rather than one. In other words, we can reinterpret the symbol "D" to be a set variable, rather than a set constant, and permit it to take two values, D₁ and D₂. D₁ we can take to be the set of things and D₂ we can take to be the set of people, with the value of D determining the surface realization of the simple quantification in which it occurs.

In accordance with the intuition about (26) and (28) that we discussed above, the choice of D₁ or D₂ can be made to depend on the quantified wff in the simple quantification. Presumably, the predicate "Linguist" would be listed in the lexicon (or somewhere else) as [+Human] (or some equivalent notation) and the predicate "Provable" would be listed as [-Human]. A redundancy rule would assign D₂ to a predicate marked [+Human] and D₁ to a predicate marked [-Human], automatically making (25) the surface form of (26) and (27) the surface form of (28). Other rules would also be necessary for more complicated wffs, but these would parallel the formation rules for semantic representations.

Let us examine this intuitively plausible proposal in more detail to see if it works out technically. We saw in IV,1,1 that the quantificational relation expressed by *some* is

(29) (1;≠,Comp₁,n₂₃,Pair₁).

Through similar reasoning we can see that the quantificational relation expressed by *no* is obtained by replacing "≠" in (29) with "=" to get

(30) (1;=,Comp₁,n₂₃,Pair₁).

Since "Linguist(x)" is [+Human], combining it with (29), as indicated by (28), gives us

(31) (≠,Comp₁,n₂₃,Pair₁)(D₂,Linguist(x)_f^x).
_{<D₂,R>}

D_2 appears in (31) precisely because "Linguist(x)" is [+Human]. From (31) we can construct the following derivation:

$$(\neq, \text{Comp}_1, n_{23})(D_2, D_2, \text{Linguist}(x)_f^x)_{\langle D_2 R \rangle}$$

$$(\neq, \text{Comp}_1)(D_2, D_2 \cap \text{Linguist}(x)_f^x)_{\langle D_2 R \rangle}$$

$$(\neq, \text{Comp } D_2, D_2 \cap \text{Linguist}(x)_f^x)_{\langle D_2 R \rangle}$$

$$\text{Comp } D_2 \neq D_2 \cap \text{Linguist}(x)_f^x_{\langle D_2 R \rangle}$$

$$(32) \text{ Comp } D_2 \neq D_2 \cap E[f_y^x \text{ satisfies Linguist}(x) \text{ in } \langle D_2 R \rangle]$$

The symbol " E " appears in (32), rather than the " E_x " that we have seen before, simply because " x " is already used as an object-language argument of "Linguist". Since the old " α " has become " x " in (28), the old " x " must become " y ".

A serious problem becomes apparent in formula (32), when we focus our attention on the set $\text{Comp } D_2$. The complement function originally appeared in R_{some} in IV, 1, 1 as a result of the appearance of the empty set λ in its non-explicit form. Since we need Comp anyway for other quantificational relations and since D is automatically the first argument of any quantificational relation, we could avoid introducing a new symbol " λ " by writing it as " $\text{Comp } D$ ". When we let D take on two values, however, $\text{Comp } D$ is no longer always the same as λ . $\text{Comp } D_1$ is still λ , if we let D_1 be the set of all things that exist at all, but it will not be λ , if we take D_1 to be the set of non-human things, as we suggested above. In the latter case, $\text{Comp } D_1$ would be the same as D_2 . $\text{Comp } D_2$, similarly, is not λ , if we take D_2 to be the set of people, as we want to. It is simply the set of non-human things.

It follows that (32) does not say what we want it to. If $\text{Comp } D_2$ is not λ , but the set of non-human things, then (32) cannot possibly be false and the corresponding formula

$$(33) \text{ Comp } D_2 = D_2 \cap E[f_y^x \text{ satisfies Linguist}(x) \text{ in } \langle D_2 R \rangle]$$

for no , which we can derive in a similar way from (30), cannot possibly be true. Formula (32) says that the set of people who are linguists is not the same as the set of non-people. This cannot be false, even if there are no linguists at all, because

there are non-people. Similarly, (33) says that the set of people who are linguists is the same as the set of non-people, a statement that cannot possibly be true, because people are not non-people, whether they are linguists or not. The problem is that we want (32) to say that the set of people who are linguists is not the same as the empty set, not that it is not the same as the non-people set, and we want (33) to say that the set of people who are linguists is the empty set. $\text{Comp } D_2$, however, is not the empty set.

We can get (32) and (33) to say what we want them to by making a slight modification in the quantificational relations from which we derived them. The set of people who are linguists cannot possibly be the same as the set of non-people, but it can be a subset of that set, if it is itself empty. This is important, because what we want (33) to say is precisely that the set of people who are linguists is empty. In other words, we can get (10) to say that the set of people who are linguists is empty by formulating it to say that that set is a subset of the set of non-people. Formally, what we have to do is to replace inequality and equality in (32) and (33), respectively, with non-superset and superset, respectively. We want (32) to have " $\not\supset$ ", instead of " \neq ", and we want (33) to have " \supset " instead of " $=$ ".

We can get (32) and (33) into this form by replacing " \neq " in (32) with " $\not\supset_{21}$ " and " $=$ " in (33) with " \supset_{21} ". We use these symbols, rather than superset symbols, because we already need the subset relation and the subscript device, so we might as well use them, rather than introducing a new relation and symbol into our theory. What we get is

$$(34) (1; \not\supset_{21}, \text{Comp}_1, n_{23}, \text{Pair}_1)$$

instead of (29), as the explicit form of R_{some} and

$$(35) (1; \supset_{21}, \text{Comp}_1, n_{23}, \text{Pair}_1)$$

instead of (30) as the explicit form of R_{no} .

In place of (32) we get

$$(36) D_2 \cap E[f_y^x \text{ satisfies Linguist}(x) \text{ in } \langle D_2 R \rangle] \not\supset \text{Comp } D_2$$

and in place of (33) we get

$$(37) D_2 \cap E[f_y^x \text{ satisfies Linguist}(x) \text{ in } \langle D_2 R \rangle] \supset \text{Comp } D_2$$

Formula (36) says that the set of people who are linguists is not a subset of the set of non-people, a statement that is true if and only if the set of people who are linguists is not the

empty set. This is exactly what we want it to say, as a semantic analysis of (28). Formula (37) says that the set of people who are linguists is a subset of the set of non-people, a statement that is true if and only if the set of people who are linguists is the empty set. This is exactly what we want it to say as a semantic analysis of

(No x) Linguist(x)

the null analog of (28).

We see that our proposal for distinguishing the two varieties of simple quantification exemplified in (25) and (27) by having our theory recognize two distinct universal domains works. We had to modify slightly two of the quantificational relations that we had constructed on the assumption that D was single-valued, but no fundamental problems seem to arise. We noted that D_1 could be taken to be either the set of non-human individuals or things or the set of all individuals, human or otherwise. Since the former set can easily be denoted by "Comp- D_2 ", we might as well take the latter choice as our interpretation of D_1 , with [-Human] still a feature determining choice of D_1 .

We developed the variable interpretation of D to distinguish between *-thing* and *-one*, but we can use it more generally to distinguish the other varieties of simple quantification that we saw in 1,1,4 as well. All we need to do to account for the simple time and place quantifications that we examined in that section is to introduce two more universal domains. D_3 , a conceptual equivalent of which is used by Prior (1957) to develop a system of tense-logic, can be interpreted as the set of times and D_4 can be interpreted as the set of places, just as D_2 is interpreted as the set of people and D_1 is the set of things. We will see in our final chapter that the seemingly entirely different category of modal operators can also be accounted for in our theory simply by introducing another universal domain, D_5 .

Section 2: Simple and Relativized Quantifiers

Our theory of quantificational relations also provides us with a framework within which to understand some of the intuitive facts about relativization that we saw in 1,1,4. What we said there was that a quantification is simple, if it involves one wff (or propositional function), and relativized, if it involves two wffs. Later, in 11,4, we said that a quantification is generalized, if it involves three or more wffs. We can make

these intuitive notions more precise by turning them into criteria for classifying quantifiers, rather than surface quantifications, as follows:

Definition 10 (Simple, Relativized, and Generalized Quantifiers): Let Q be a quantifier and let $n = \text{order}(Q)$. Q is said to be

- (a) simple, if $n=1$;
- (b) relativized, if $n=2$;
- (c) generalized, if $n>3$.

In 11,3,2 we saw that *some*, *no*, and the numerals are the only quantifiers that occur in English whose members are of order 1, that the members of *all* are of order 2, and that the elementary quantifiers that belong to *the* are of order 3. It follows, according to Definition 10, that *some*, *no*, and the numerals are the only English quantifiers that are simple, that *all* is a relativized quantifier, and that *the* is generalized.

Given Definition 10, we can begin to make some sense, formally speaking, of the intuitive discussion of relativization that we had in 1,1,4. First we need another definition, as follows:

Definition 11 (Simple Occurrence of Relativized Quantifier): Let Q be a relativized quantifier and let

(38) $(Q\alpha)(\Psi;\Phi)$

be a formula (semantic representation) that contains Q. Formula (38) is said to be a simple occurrence of Q if " Ψ " is of the form

(39) $\alpha \in D$.

In such a case we also write

(40) $(Q\alpha)\Phi$

as an alternate (equivalent) form of (38) and we say that (40) is a simple occurrence of Q.

Simple occurrences of relativized quantifiers, in other words, are a device for making relativized quantifiers, in the sense of Definition 10, look like simple ones, in the same sense, when

the first quantified wff is general.

It makes sense to write (38) as (40), when " Ψ " is of the form (39), because of what happens when we take " Ψ " to be (39) under the standard set assignment. Suppose we take *most* as our example and replace " Ψ " in (23) with (39). This gives us the formula

$$(41) \quad K(E[x \in D \text{ and } f_x^\alpha \text{ satisfies } \alpha \in D \text{ in } \langle DR \rangle \text{ and } f_x^\alpha \text{ satisfies } \Phi \text{ in } \langle DR \rangle]) > K(E[x \in D \text{ and } f_x^\alpha \text{ satisfies } \alpha \in D \text{ in } \langle DR \rangle \text{ and } f_x^\alpha \text{ satisfies } \neg \Phi \text{ in } \langle DR \rangle])$$

as the truth condition of

$$(42) \quad (\text{Most } \alpha)(\Psi; \Phi)$$

when " Ψ " is (39). The assignment function f_x^α satisfies the formula " $\alpha \in D$ ", however, if and only if x is a member of D , so the statement

$$f_x^\alpha \text{ satisfies } \alpha \in D \text{ in } \langle DR \rangle$$

says the same thing as

$$x \in D$$

This means that (41) says the same thing as

$$K(E[x \in D \text{ and } x \in D \text{ and } f_x^\alpha \text{ satisfies } \Phi \text{ in } \langle DR \rangle]) > K(E[x \in D \text{ and } x \in D \text{ and } f_x^\alpha \text{ satisfies } \neg \Phi \text{ in } \langle DR \rangle])$$

which itself reduces to

$$(43) \quad K(E[x \in D \text{ and } f_x^\alpha \text{ satisfies } \Phi \text{ in } \langle DR \rangle]) > K(E[x \in D \text{ and } f_x^\alpha \text{ satisfies } \neg \Phi \text{ in } \langle DR \rangle])$$

because the second " $x \in D$ " is superfluous. Formula (43), however, is exactly what we found the truth condition of *most* to be in 11,2,2, when we naively took it to be a simple quantifier.

We see that a formula like

$$(44) \quad (\text{Most } \alpha)\Phi$$

is really an abbreviation for (42) under the conditions specified in Definition 11. Relativized quantifiers can have simple

occurrences, but they are still semantically relativized. The genuinely simple quantifiers, like *some* and *no*, are genuinely simple for deeper semantic reasons that we have already examined in great detail and summarized in Definition 10.

Definition 11 enables us to explain the fact, noted in 1,1,4 and 11,3,2, that *only* can occur only relativized. All we need to do is to apply the definition to the presupposition and assertion of *only* and see what happens. The presupposition of

$$(45) \quad (\text{Only } \alpha)(\Psi; \Phi)$$

as we saw in 11(100), is the relativized existential quantification

$$(46) \quad (\text{Some } \alpha)(\Psi; \Phi)$$

and the assertion of (45) is the relativized universal quantification

$$(47) \quad (\text{All } \alpha)(\Phi; \Psi)$$

The set-theoretic relation expressed by (46) is given by

$$(48) \quad D \cap \Psi_f^\alpha \cap \Phi_f^\alpha \neq \lambda$$

as we saw in 111(46), and the quantificational relation expressed by (47) is given by

$$(49) \quad D \cap \Phi_f^\alpha \subseteq \Psi_f^\alpha$$

as we saw in 111(48). Technically, we should use a slightly modified version of (48), as we discussed this in our last section, but (48) will suffice for our present purposes.

Definition 11 tells us that any simple occurrence

$$(50) \quad (\text{Only } \alpha)\Phi$$

of *only* is semantically of the form (45) with

$$(51) \quad \Psi = (\alpha \in D).$$

It follows that the presupposition and assertion, respectively, of (45) are of the form (46) and (47), with the stipulation that (51) holds. This means that the set-theoretic relation (48) of the presupposition becomes

$$D \cap (\alpha \in D)_f^\alpha \cap \phi_f^\alpha \neq \lambda$$

which reduces, first, to

$$D \cap D \cap \phi_f^\alpha \neq \lambda$$

and then to

$$(52) \quad D \cap \phi_f^\alpha \neq \lambda$$

It also means that the quantificational relation (49) expressed by the assertion becomes

$$D \cap \phi_f^\alpha \subseteq (\alpha \in D)_f^\alpha$$

which reduces to

$$(53) \quad D \cap \phi_f^\alpha \subseteq D$$

Formulas (52) and (53) give us, respectively, the set-theoretic relations expressed by the presupposition and assertion of (50).

Formula (52) presents no problem, because it gives us the quantificational relation expressed by the simple existential quantification

$$(54) \quad (\text{Some } \alpha) \phi$$

All it says is that there are some things that have the property ϕ and this can be either true or false.

Formula (53), however, is peculiar, because it can never be false. What (53) says is that the individuals that have the property ϕ are individuals, something that we knew anyway, whether or not there even are any such individuals. The point is that the intersection of D with any set, even the empty set, is a subset of D , so there would never be any point, other than a pedagogical or expository one, of asserting that this is true of a particular set.

Getting back to English, this explains why the sentence

$$(55) \quad \text{*Only things are provable.}$$

which we saw in 1,1,4 and 11,3,2, is semantically anomalous. The presupposition of (55) is

$$(56) \quad \text{Some things are provable.}$$

a surface realization of (52), and its assertion is

$$(57) \quad \text{*All provable things are things.}$$

a surface realization of (53). Sentence (56) is perfectly normal, but (57) tells us nothing we did not know already. Sentence (55) is not semantically anomalous because it is meaningless or self-contradictory. It is true if there is even one provable thing and it "lacks truth value" if there are no provable things, just like any other sentence whose presupposition fails to hold. The reason (55) is peculiar is that there are no conceivable circumstances under which it would be false. For our purposes, the most significant aspect of this situation is the fact that it follows directly from one of the most fundamental notions of set theory, the fact that the universal domain includes everything that there is.

The two definitions of this section also enable us to make some sense of the comparative facts of quantification in Greek, Latin, German, and English that we discussed intuitively in 1,1. Semantically, we have seen that *some* and *no* are simple quantifiers, unlike nearly all of the other quantifiers we have discussed, because of the normal-form properties of their set-theoretic relations. We can now see that it is this fact that underlies the special treatment given to *some* and *no* in Latin and Greek. As we saw in 1,1,2, both languages treat the relativized quantifiers in more or less the same way, as adjectives, but they both provide special treatment for each of the two semantically simple quantifiers *some* and *no*. Some numbers also receive special treatment in both languages, but the numerical quantifiers are also easily shown to be simple, just like *some* and *no*. Latin generalized its treatment of *some* and *no* somewhat by using it also for a slightly broader class of quantifiers, such as *other*, which are closely related to *some* in meaning, but this does not alter the basic contrast with its treatment of the general class of relativized quantifiers.

In our intuitive discussion of English and German quantification in 1,1,4 we pointed out that quantifications in those languages that involve the classical quantifiers can collapse their quantifiers and their first quantified property into a single word, if the quantifications are simple. Not surprisingly, at this point, two of the collapsible quantifiers turn out to be the semantically simple ones *some* and *no*. The third collapsible quantifier is the relativized quantifier *all*, which is inter-definable with these two simple quantifiers in the sense that we discussed intuitively in 1,2,1 and that we will examine in depth in the next chapter. No other relativized quantifiers are

collapsible in this way, so the collapsibility of *all* can be viewed as being attributable to the pressure for surface uniformity brought to bear by this deeper semantic interdefinability.

Section 3: Reducibility

Although both processes involve the apparent transformation of relativized quantifiers into simple ones, the phenomenon of reducibility that we examined in II is very different from the simplification phenomenon that we analyzed in the last section. A simple occurrence of a relativized quantifier still contains the second wff as a quantified wff. Formula (42) turns into (44) with ϕ quantified in both. The first wff, Ψ , disappears, because, in terms of any real contribution to the meaning of the sentence, it was never really there. As an instance of " $\alpha=D$ ", it adds no information to (42) that could not be expressed in (44), since D itself is already an argument of any set-theoretic relation.

Reducibility, in contrast, replaces the two wffs with some truth-functional combination of them and it places no restrictions on the form of Ψ . Whether or not a relativized quantification is reducible to a simple occurrence is a property of the quantifier itself, not of the form of the wffs it operates on. As we saw in II, the universal quantifier is reducible, but *most* is not, regardless of the form of Ψ .

We can formalize these facts as follows:

Definition 12 (Reducibility and Reduced Form of Relativized

Quantifier): Let Q be a relativized quantifier. Q is said to be reducible or reducible to its simple form if there is a truth-functional mapping t from R^2 into R such that, for every Ψ , $\phi \in R$, the simple occurrence

$$(58) \quad (Q\alpha) t(\Psi, \phi)$$

is equivalent to the formula

$$(59) \quad (Q\alpha)(\Psi; \phi).$$

The important thing to realize about this definition is that (58) is a simple occurrence, as defined in Definition 11, because $t(\Psi, \phi)$ is a single member of R . This contrasts with the explicitly relativized quantification (59), in which $Q\alpha$ operates on the ordered pair (Ψ, ϕ) of members of R . This also explains, of course, the appearance of a comma in (58), instead of the semi-colon of (59). As long as we specify that we are dealing only

with normal forms, we could get by with commas in formulas like (59), but we will retain the semi-colon for clarity.

The fact that some relativized quantifiers are reducible and others are not is easily explained in terms of their quantificational relations. The quantificational relation expressed by the formula

$$(60) \quad (All \alpha)(\Psi; \phi)$$

for example, is given by

$$(61) \quad D \cap \Psi_f^\alpha \subseteq \phi_f^\alpha \\ <DR> <DR>$$

as we saw in connection with (22). It follows that the set-theoretic relation expressed by any simple occurrence

$$(62) \quad (All \alpha)\Gamma$$

is given by

$$(63) \quad D \subseteq \Gamma_f^\alpha \\ <DR>$$

because (62) is just an abbreviation for

$$(64) \quad (All \alpha)(\alpha \in D; \Gamma)$$

as specified in Definition 11, and $s^*(\alpha=D)=D$. What Definition 12 says is that *all* is reducible, if there is a truth-functional mapping t such that

$$(65) \quad \Gamma = t(\Psi, \phi)$$

and (62) is equivalent to (60) for that Γ .

One of the most useful properties of the standard set assignment, provable directly from Definition 5, is the set of relations

$$(66) \quad s^*A = \cap(s^*, s^*)$$

$$s^*V = \cup(s^*, s^*)$$

$$s^*- = \text{Comp } s^*$$

$$s^*(\text{truth}) = D$$

$$s^*(\text{falsity}) = \lambda$$

and their inverses. In other words, the extension of a conjunction is the intersection of the extensions of the conjuncts, and so on. We have already used the third relation in (66), when we took III(38) to be the same as III(39) in III,2,2. It follows directly from (66) that, if we can find a set γ that is expressible entirely in terms of Ψ_f^α , Φ_f^α , D , λ , n , u , and Comp , then we can take

$$(67) \quad s^*(\Gamma) = \gamma$$

so

$$(68) \quad \Gamma = s^{*-1}(\gamma).$$

The inverses of (66) then give us t .

With a little reflection, we can see that formula (61) is equivalent to the formula

$$(69) \quad D \subseteq (\text{Comp } \Psi_f^\alpha) \cup \Phi_f^\alpha$$

The equivalence is easy to illustrate in terms of Venn diagrams and just as easy to prove from the basic properties of the set-theoretic operators involved. If we now take

$$(70) \quad \gamma = (\text{Comp } \Psi_f^\alpha) \cup \Phi_f^\alpha$$

and apply (68), then we get

$$(71) \quad \Gamma = s^{*-1}((\text{Comp } \Psi_f^\alpha) \cup \Phi_f^\alpha).$$

Applying (66) to (71) gives us, first,

$$\Gamma = s^{*-1}(\text{Comp } \Psi_f^\alpha) \vee s^{*-1}(\Phi_f^\alpha)$$

and then

$$(72) \quad \Gamma = -\Psi \vee \Phi.$$

The disjunction in (72), however, can be rewritten as a material implication, giving us

$$(73) \quad \Gamma = \Psi \supset \Phi$$

and

$$t = \supset.$$

Comparing (69) to (63) and combining (73) with (62) gives us the result that (60) is equivalent to, and thus reducible, in the sense of Definition 12, to

$$(74) \quad (\text{All } \alpha)(\Psi \supset \Phi).$$

This is exactly the result that we got intuitively in II,1,4.

As an example of a non-reducible quantifier we can take *most*. If *most* is reducible, then we should be able to find a truth-functional mapping t such that the relativized quantification expressed by

$$(75) \quad (\text{Most } \alpha)(\Psi; \Phi)$$

is equivalent to the simple occurrence

$$(76) \quad (\text{Most } \alpha) t(\Psi, \Phi).$$

The quantificational relation expressed by (75), as we have already seen, is given by

$$(77) \quad K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) > K(D \cap \Psi_f^\alpha \cap \text{Comp } \Phi_f^\alpha).$$

If we take Γ as in (65), then the set-theoretic relation expressed by (76) is given by

$$(78) \quad K(D \cap \Gamma_f^\alpha) > K(D \cap \text{Comp } \Gamma_f^\alpha)$$

as we can see from (43). It follows that *most* is reducible, if (78) is equivalent to (77) for some choice of t in (65).

It is not difficult to see that there is no such t . Formula (66) tells us that (77) is equivalent to

$$(79) \quad K(D \cap (\Psi \wedge \Phi)_f^\alpha) > K(D \cap (\Psi \wedge -\Phi)_f^\alpha)$$

and that (78) is equivalent to

$$(80) \quad K(D \cap \Gamma_f^\alpha) > K(D \cap (-\Gamma)_f^\alpha).$$

Formulas (79) and (80) can be equivalent only if the formulas

$$(81) \quad \Gamma = \Psi \wedge \Phi$$

$$(82) \quad -\Gamma = \Psi \wedge -\Phi$$

both hold. From (81), however, we can derive the formula

$$-\Gamma = -\Psi \vee -\Phi$$

which is incompatible with (82), if we expect both equations to be identities, true for all Ψ and Φ . It follows that there is no t that will make (77) and (78) equivalent via (65) and thus that *most* is not reducible.

It is worth pointing out that the question of non-reducibility vs. reducibility does not depend on the explicit forms of the quantificational relations, but on their interaction with the values of s^* . It is the set-theoretic content of the relations, not their form, on which reducibility depends. The explicit form of simplified *all* in (63) is just the relation

$$(83) \quad (1; \subseteq)$$

which is almost the same as the explicit form of relativized *all* that we saw in (5). The explicit form of simplified *most*, as given in (78), is the relation

$$(84) \quad (1; >, K, n_{23}, n_{12}, \text{Comp}_4, \text{Pair}_{12})$$

which is exactly the same as relativized *most* in (16), except for the initial "order" number and the subscripts. Neither (83) nor (84) gives us any clue from their form alone as to the possible reducibility of their relativized versions.

CHAPTER 3: THE GROUP STRUCTURE OF QUANTIFICATION

Section 1: Quantificational Relations and Interdefinability

In 1.2.1 we examined the following relations that hold among the three classical quantifiers:

$$(85) \quad \text{some} = \text{not no} = \text{not all not}$$

$$\text{not some} = \text{no} = \text{all not}$$

$$\text{not some not} = \text{no not} = \text{all}$$

We can gain a better understanding of these relations by examining the way in which they reflect the quantificational relations of their quantifiers. To begin with, we have to equalize the number of set arguments involved, because *some* and *no* are of order 1, but *all* is of order 2. This means that the relations in (85) hold strictly of the elementary quantifiers that are members of the quantifiers involved, rather than the quantifiers themselves. If we take care to interpret (85) in this way, then we can still treat the relations as holding of the quantifiers, as they would in the case of quantifiers of equal orders.

What we want to do is to compare the set-theoretic relations expressed by the members of *all*, *some*, and *no* to see if the relations in (85) can be seen in them in some form. We get the same results, whether we compare simplified *all* with *some* and *no* or relativized *some* and *no* with *all*, so we will consider only the former case.

As we saw in IV,2.1, the quantificational relation expressed by *some* is

$$(86) \quad R_{\text{some}} = (1; \not\subseteq_{21}, \text{Comp}_1, n_{23}, \text{Pair}_1)$$

and the quantificational relation expressed by *no* is

$$(87) \quad R_{\text{no}} = (1; \subseteq_{21}, \text{Comp}_1, n_{23}, \text{Pair}_1)$$

Comparing (86) and (87) makes their relationship obvious. The only difference between the two relations is that (86) has a " $\not\subseteq$ ", where (87) has a " \subseteq ". Apparently the fact that

$$\text{some} = \text{not no}$$

and

$$\text{no} = \text{not some}$$

is reflected in the fact that the first relational component of each relation is the negation of that of the other.

As we saw in IV,1,1, the quantificational relation expressed by *all* is

$$(88) R_{all}^* = (2; \subseteq, n_{12}^*)$$

This means that the formula

$$(89) (All \alpha)(\Psi; \Phi)$$

is true if and only if the relation (88) holds of the two sets $s^*(\Psi)$ and $s^*(\Phi)$, in that order, preceded by the set argument D . To compare (88) properly with (86) and (87) we must first transform it into the set-theoretic relation expressed by a simple occurrence

$$(90) (All \alpha)\Phi.$$

As we saw in IV,2,2, (90) is really of the form

$$(91) (All \alpha)(\alpha \in D; \Phi)$$

which is true if and only if (88) holds of the sets D , D , and $s^*(\Phi)$.

If we actually apply (88) to these sets, we get, in sequence,

$$(\subseteq, n_{12}^*)(D, D, \Phi_f^\alpha)$$

$$(\subseteq)(D \cap D, \Phi_f^\alpha)$$

$$(92) D \cap D \subseteq \Phi_f^\alpha$$

$$D \subseteq \Phi_f^\alpha$$

$$(\subseteq)(D, \Phi_f^\alpha)$$

This gives us

$$(93) R_{all}^1 = (1; \subseteq)$$

where the 1's indicate one less than the number of set arguments involved, not the order, as the set-theoretic relation expressed by simplified *all*.

A direct comparison of (93) with (86) and (87) tells us nothing about the origin of (85). There seems, on the surface, to be no relation between the form of (93) and the forms of the other two relations. We can transform (93) into a more revealing form, however, by applying a little set algebra. According to (69), (92) is equivalent to the formula

$$(94) D \subseteq (Comp D) \cup \Phi_f^\alpha$$

Since taking *Comp*'s reverses the order of the subset relation and turns unions into intersections, formula (94), and therefore formula (92), is equivalent, in sequence to

$$Comp D \supseteq Comp((Comp D) \cup \Phi_f^\alpha)$$

$$Comp D \supseteq (Comp Comp D) \cap Comp \Phi_f^\alpha$$

$$(95) Comp D \supseteq D \cap Comp \Phi_f^\alpha$$

Putting (95) into explicit form, we get, in sequence

$$(\subseteq_{21})(Comp D, D \cap Comp \Phi_f^\alpha)$$

$$(\subseteq_{21}, Comp_1)(D, D \cap Comp \Phi_f^\alpha)$$

$$(\subseteq_{21}, Comp_1, n_{23})(D, D, Comp \Phi_f^\alpha)$$

$$(\subseteq_{21}, Comp_1, n_{23}, Pair_1)(D, Comp \Phi_f^\alpha)$$

$$(96) (\subseteq_{21}, Comp_1, n_{23}, Pair_1, Comp_2)(D, \Phi_f^\alpha)$$

From (96) we get

$$(97) R_{all}^1 = (1; \subseteq_{21}, Comp_1, n_{23}, Pair_1, Comp_2)$$

as an equivalent form of (93) for the set-theoretic relation expressed by simple *all*.

Formula (97) is significantly more complicated in its formulation than (93) and is thus less useful for some purposes. As the lexical entry of *all*, for example, we would most likely choose the relativized version of (93), rather than that of (97), because of the former's greater simplicity. For our present purposes, however, (93) is useless, as we have seen, but (97) is exactly what we need. Formula (97) is exactly the same as formula (87), except for the additional final component, Comp_2 . What this shows is that the equivalence of *no* and *all not*, as expressed in (85), is a reflection of the final Comp_2 in the set-theoretic relation expressed by *all*. This result should not be surprising in view of the relations contained in (66).

The other interdefinability relationships we have discussed are reflected set-theoretically in exactly the same way. In 1,2,1, for example, we saw Altham's relationships

$$(98) \text{ many} = \text{not few} = \text{not nearly all not}$$

$$\text{not many} = \text{few} = \text{nearly all not}$$

$$\text{not many not} = \text{few not} = \text{nearly all.}$$

For these quantifiers there is no need to equalize the number of set arguments, because all three of them are of order 2.

As we saw in IV,1,2, the quantificational relation expressed by *many* is

$$(99) R_{\text{many}} = (2; \geq n, K, n_{123}).$$

According to the analysis that we just made of the classical quantifiers, the quantificational relation expressed by *few*, since it is equivalent, according to (98), to *not many*, should be the same as (99), except that its initial relational component should be the negation of $\geq n$. This gives us

$$(100) R_{\text{few}} = (2; < n, K, n_{123})$$

as the quantificational relation expressed by *few*.

If we expand (100) in the usual way, we get the derivation

$$(< n, K, n_{123}) (D, \Psi_f^\alpha, \Phi_f^\alpha) \\ < DR > < DR >$$

$$(< n, K) (D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) \\ < DR > < DR >$$

$$(< n) (K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) \\ < DR > < DR >)$$

$$K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) < n \\ < DR > < DR >$$

$$K(D \cap \bigcup_x [f_x^\alpha \text{ satisfies } \Psi \text{ in } < DR >]$$

$$\cap \bigcup_x [f_x^\alpha \text{ satisfies } \Phi \text{ in } < DR >]) < n$$

$$(101) K(\bigcup_x [x \in D \text{ and } f_x^\alpha \text{ satisfies } \Psi \text{ in } < DR >$$

$$\text{and } f_x^\alpha \text{ satisfies } \Phi \text{ in } < DR >]) < n$$

Formula (101) is clearly just a Kaplan-style notational variant of 11(51) and thus is exactly the analysis of *few* that we want. We got it by applying a relationship we discovered to hold in the case of *some* and *no* to the quantificational relations of the analogous quantifiers *many* and *few*. Again we see that Kaplan's notational framework follows directly from our underlying set-theoretic theory of quantification.

Since *nearly all* is equivalent to *few not*, according to (98), our analysis of *all* and *no* suggests that the quantificational relation expressed by *nearly all* should be

$$(102) R_{\text{nearly all}} = (2; < n, K, n_{123}, \text{Comp}_3)$$

Expanding (102) gives us the derivation

$$(< n, K, n_{123}, \text{Comp}_3) (D, \Psi_f^\alpha, \Phi_f^\alpha) \\ < DR > < DR >$$

$$(< n, K, n_{123}) (D, \Psi_f^\alpha, \text{Comp } \Phi_f^\alpha) \\ < DR > < DR >$$

$$(< n, K) (D \cap \Psi_f^\alpha \cap \text{Comp } \Phi_f^\alpha) \\ < DR > < DR >$$

$$(< n) (K(D \cap \Psi_f^\alpha \cap \text{Comp } \Phi_f^\alpha)) \\ < DR > < DR >$$

$$K(D \cap \Psi_f^\alpha \cap \text{Comp } \Phi_f^\alpha) < n \\ < DR > < DR >$$

$$K(D \cap \bigcup_x [f_x^\alpha \text{ satisfies } \Psi \text{ in } < DR >$$

$$\cap \bigcup_x [f_x^\alpha \text{ satisfies } \neg \Phi \text{ in } < DR >]) < n$$

$$(103) K(\bigcup_x [x \in D \text{ and } f_x^\alpha \text{ satisfies } \Psi \text{ in } < DR > \text{ and}$$

$$f_x^\alpha \text{ satisfies } \neg \Phi \text{ in } < DR >]) < n.$$

Formula (103) says that nearly every Ψ has Φ if there are fewer than n Ψ 's that lack Φ . In other words, there is no set of n Ψ 's, every one of which has $\neg\Phi$. It follows that in every set of n Ψ 's there is at least one that has Φ . This, of course, is exactly the analysis of *nearly all* that is contained in 11(50) and, again, we obtained it by applying our analysis of the set-theoretic content of interdefinability in the classical case to a different quantifier.

Section 2: Outer, Inner, and Dual Negation

The results of the preceding section can be generalized in the following way:

Definition 13 (Outer, Inner, and Dual Negation): Let Q be a quantifier and let the quantificational relation expressed by Q be given by

$$(104) \quad R_Q = (n; R_1, R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k)$$

A quantifier Q^+ is said to be the outer negation of Q if there is an elementary quantifier q^+ in Q^+ such that the set-theoretic relation expressed by q^+ is given by

$$(105) \quad R_{q^+} = (n; \neg R_1, R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k)$$

A quantifier Q^* is said to be the inner negation of Q if there is an elementary quantifier q^* in Q^* such that the set-theoretic relation expressed by q^* is given by

$$(106) \quad R_{q^*} = (n; R_1, R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k, \text{Comp}_{n+1})$$

A quantifier Q' is said to be the dual negation of Q if there is an elementary quantifier q' in Q' such that the set-theoretic relation expressed by q' is given by

$$(107) \quad R_{q'} = (n; \neg R_1, R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k, \text{Comp}_{n+1})$$

The set whose members are Q^+ , Q^* , and Q' is said to be the negation set of Q .

We have to formulate this definition in terms of the set-theoretic relations expressed by elementary quantifiers, rather than the quantificational relations expressed by the quantifiers themselves, because of the possibility that the quantifiers may have different orders. We have already seen that this possibility is realized in the case of the classical quantifiers.

The fact that each of the relations

$$(108) \quad Q^+ = \neg Q$$

$$Q^* = Q^-$$

$$Q' = \neg Q^-$$

holds, as in (85) and (98), follows directly from (104), (105), (106), and (107), as do both the existence and uniqueness of the negation set for any quantifier. In fact, it was our desire to establish a set-theoretic basis for the relations in (108) that led us to introduce Definition 13 in the first place. We will soon see that (108) is useful in proving a very fundamental result about the formal structure of the class of quantifiers.

As an example of Definition 13 we can refer to our earlier discussion of *all* and *some*. As we saw in the last section, the only difference between *all* and *some* is that the first relational component of R_{all} is the negation of the first relational component of R_{some} and that R_{all}^1 has an extra final component Comp_2 . It follows that

$$all = (some)'$$

in the notation of Definition 13. Formulas (108), however, tell us that the dual negation of any quantifier, when dual negated, always produces a result that is equivalent to the quantifier itself, a fact that we can also get directly from Definition 13. This means that we also have the formula

$$some = (all)'$$

and that we should be able to express this set-theoretically as stated in the definition.

Since the quantificational relation expressed by *all* can be written as

$$R_{all} = (2; \neg, n_{12})$$

as in (88), the set-theoretic relation expressed by relativized *some* should be given, according to Definition 13, by

$$(109) \quad R_{some}^2 = (2; \neg, n_{12}, \text{Comp}_3)$$

Does this work? Expanding (109) in the usual way gives us the derivation

$$(\perp, n_{12}, \text{Comp}_3) (D, \psi_f^\alpha, \phi_f^\alpha)$$

$$(\perp, n_{12}) (D, \psi_f^\alpha, \text{Comp } \phi_f^\alpha)$$

$$(\perp) (D \cap \psi_f^\alpha, \text{Comp } \phi_f^\alpha)$$

$$(110) \quad D \cap \psi_f^\alpha \not\subseteq \text{Comp } \phi_f^\alpha$$

Intuitively, formula (110) says that "(Some α)($\psi; \phi$)" is true if and only if the set of individuals that have ψ is not entirely contained in the set of individuals that lack ϕ . This means that there is an individual that has ψ that has ϕ as well. Formally, it is not difficult to show that (110) is equivalent to the formula

$$D \cap \psi_f^\alpha \cap \text{Comp } \phi_f^\alpha \neq \lambda$$

which, as we already saw in III(46), gives us the set-theoretic relation expressed by relativized *some*. We see that our set-theoretic definition of dual negation works in either direction, whether we take Q to be *some* or we take it to be *all*.

The following result also follows directly from Definition 13, but we will prove it from (108) because of its more compact formulation:

Lemma 1: If Q is a quantifier, then the following relations hold:

$$(a) \quad (Q')' = (Q^*)^* = (Q^+)^+ = Q$$

$$(b) \quad (Q^*)' = (Q')^* = Q^+$$

$$(c) \quad (Q^+)' = (Q^*)^+ = Q'$$

$$(d) \quad (Q')^+ = (Q^+)' = Q^*$$

Proof: From (108) we get each of the following relations:

$$(a) \quad (Q')' = -(Q')- = -(-Q-)- = Q$$

$$(Q^*)^* = (Q^*)- = (Q-)- = Q$$

$$(Q^+)^+ = -(Q^+)- = -(-Q) = Q$$

$$(b) \quad (Q^*)' = -(Q^*)- = -(Q-)- = -Q = Q^+$$

$$(Q')^* = (Q')- = (-Q-)- = -Q = Q^+$$

$$(c) \quad (Q^+)' = -(Q^+)- = -(Q-)- = -Q- = Q'$$

$$(Q^+)^* = (Q^+)- = (-Q-)- = -Q- = Q'$$

$$(d) \quad (Q')^+ = -(Q') = -(Q') = -(-Q-) = Q- = Q^*$$

$$(Q^+)' = -(Q^+)- = -(-Q)- = Q- = Q^*$$

Q.E.D.

The fact that two immediately successive applications of "--" yields the identity function follows directly from the set-theoretic character of our quantifiers. Some languages or dialects use double negation in their surface grammar to express negative concord, but considerations of simplicity and economy would argue against permitting such a device on the level of semantic representations. This argument justifies the reasoning of (a), (b), (c), and (d).

The structure described in the lemma is not an unfamiliar one. If we replace the symbol " Q " with the non-zero real variable " x " and interpret "+", "*", and "-" as "additive inverse", "multiplicative inverse", and "negative reciprocal", respectively, rather than as outer, inner, and dual negation, then the relations of the lemma turn into the identities

$$(a) \quad -\frac{1}{-\frac{1}{x}} = (x^{-1})^{-1} = -(-x) = x$$

$$(b) \quad -\frac{1}{x^{-1}} = (-\frac{1}{x})^{-1} = -x$$

$$(c) \quad -(x^{-1}) = (-x)^{-1} = -\frac{1}{x}$$

$$(d) \quad -(-\frac{1}{x}) = -\frac{1}{-x} = \frac{1}{x}$$

In fact, the structure in the lemma is quite common. If we take Q to be a musical theme or melody, for example, and interpret "+", "*", and "-", respectively, as inversion, retrograde, and retrograde inversion, then all of the relations of the lemma hold. This shows that the structure we are dealing with is very general and suggests that we might be able to learn something about quantifiers by investigating these alternative models.

We can deepen our insight into the nature of this structure by abstracting away from the class of quantifiers and focusing on the negation (inverse) operations themselves, as follows:

Definition 14 (Negation Functions): Let g_i , $i=1,2,3$, be functions defined on the class of quantifiers (in a given language L) as follows: For all quantifiers Q ,

$$g_1(Q) = Q'$$

$$g_2(Q) = Q^*$$

$$g_3(Q) = Q^+.$$

The functions g_i are said to be the negation functions on the class of quantifiers (in L).

The function g_1 is called the dual negation

function, the function g_2 is called the inner negation function, and the function g_3 is called the outer negation function.

The need to specify the language L in Definition 14 results from the fact that we are defining functions on a class of quantifiers in contrast to Definition 13, in which we were talking about individual quantifiers, each expressible in set theory, whatever language they might be in. Functions are not well-defined, if their domains are not explicitly specified.

Lemma 1 now immediately yields the following result:

Lemma 2: If g_i , $i=1,2,3$, are the negation functions on the class of quantifiers (in L) and g_4 is the identity function on that class, then the following relations hold:

$$(a) \quad g_i \circ g_i = g_4, \quad i=1,2,3$$

$$(b) \quad g_1 \circ g_2 = g_2 \circ g_1 = g_3$$

$$(c) \quad g_2 \circ g_3 = g_3 \circ g_2 = g_1$$

$$(d) \quad g_3 \circ g_1 = g_1 \circ g_3 = g_2$$

where " \circ " is interpreted to mean function composition.

In other words, the composition of any negation function with itself is the identity function and the composition of any negation function with either other negation function is the remaining negation function. Each condition in Lemma 2 is simply a restatement of the corresponding condition in Lemma 1 directly in terms of the negation operations themselves, as characterized in Definition 14. From Lemma 2, however, we get the following further basic result:

Theorem: The set whose members are the negation functions and the identity function on the class of quantifiers in L is the elementary Abelian 2-group of order 4 with function composition as the group operation. The set of restrictions of these functions to the class of quantifiers in L which equal their own dual negations is the elementary Abelian 2-group of order 2.

It is difficult to imagine a more tightly-knit structure than the one described in Lemma 2 and identified in the Theorem. As we saw in connection with Lemma 1, this is the structure not only of the interaction of quantifiers and negation, but also of the interaction of non-zero real numbers with their various inverse operations, among other things.

CHAPTER 4: QUANTIFIERS AND MODAL LOGIC

Section 1: Modal Logic and Dual Negation

The existence of dual negations, which we examined in the last section, suggests a possible connection between quantifiers and modal logic. Prior (1957) summarizes the defining characteristics of a modal logic, first set down by Łukasiewicz, as follows:

To count as a modal logic, according to Łukasiewicz, a system must contain a pair of one-argument operators forming statements out of statements, with the following properties: The more powerful modal operator, which we may symbolize as L , must be such that Lp is a stronger form than p , and yet not so strong as to be never true. That is, 'If Lp then p ,' but not its converse must be a logical law, and the simple NLp must not be one. And the feebler modal operator, which we may symbolize as M , must be such that Mp is a weaker, more non-committal form than p , and yet not so non-committal as to be never false. That is, 'If p then Mp ,' but not its converse, must be a logical law, and the simple Mp must not be one. Finally, Mp must be equivalent to $NLNp$ and Lp to $NMNp$.

(pp. 2-3)

In the notation and terminology that we have been using, the formulas in (111) must be logically true (valid) and the formulas in (112) must not be:

- (111) (a) $L\phi \supset \phi$
 (b) $\phi \supset M\phi$
 (c) $M\phi \equiv -L-\phi$
 (d) $L\phi \equiv -M-\phi$
- (112) (a) $\phi \supset L\phi$
 (b) $-L\phi$
 (c) $M\phi \supset \phi$
 (d) $M\phi$

where L and M are mappings from R_0 into R_0 .

Prior suggests that these criteria can be generalized by permitting L and M to be mappings from R_1 into R_0 , that is, by permitting them to operate on open statements, rather than restricting them to the closed statements that represent members of R_0 . He points out that, "if, however, this extension of the notion of a statement is to be permitted when applying Łukasiewicz's definition of 'modal', we must also count the ordinary theory of quantification as a modal system" (p. 6). He explains, in his own notation, that if we take L to be $All \alpha$ and M to be $Some \alpha$, then the conditions (111) and (112) are satisfied, as follows:

- (113) (a) $(All \alpha) \phi \supset \phi$
 (b) $\phi \supset (Some \alpha) \phi$
 (c) $(Some \alpha) \phi \equiv -(All \alpha) -\phi$
 (d) $(All \alpha) \phi \equiv -(Some \alpha) -\phi$
- (114) (a) $\phi \supset (All \alpha) \phi$
 (b) $-(All \alpha) \phi$
 (c) $(Some \alpha) \phi \supset \phi$
 (d) $(Some \alpha) \phi$

Each of the formulas in (113) is an instance of the corresponding formula in (111) and is logically true. Each of the formulas in (114) is an instance of the corresponding formula in (112) and is not logically true. This remains the case, even if we take these quantifiers in their most general form, as mappings from R^n into R , and replace (113) and (114) by the resulting analogs.

If the classical quantifiers *all* and *some* form a modal logic, the question naturally arises whether other quantifiers do as well. We already know that (111c) and (111d) are logically true for quantifiers M and L , if L is the dual negation of M . Taken together, all they say is that any quantifier is equivalent to the dual negation of its dual negation, a fact that we proved in the Theorem of Chapter 3. It is this fact that suggests a connection between quantifiers and modalities, but we must check the other six conditions to see whether they also hold. Can we show that any quantifier and its dual negation form a modal logic, in Prior's modified sense of Łukasiewicz?

Unfortunately, the answer to this intriguing question is "No". As a counterexample to the proposed generalization, we can let L =Nearly all α and M =Many α , so that (111c) and (111d),

respectively, become the formulas

$$(\text{Many } \alpha) \equiv -(\text{Nearly all } \alpha) - \phi$$

$$(\text{Nearly all } \alpha) = -(\text{Many } \alpha) - \phi.$$

As we have seen before, these two formulas are logically true, because *many* and *nearly all* are mutual dual negations. This is the significance, in part, of (98). Formulas (111a) and (111b), however, become, respectively,

$$(115) (\text{Nearly all } \alpha) \phi \supset \phi$$

$$(116) \phi \supset (\text{Many } \alpha) \phi$$

neither of which is logically true. If both (115) and (116) were true logically, then

$$(\text{Nearly all } \alpha) \phi \supset (\text{Many } \alpha) \phi$$

would also be. As Altham shows and as we can see intuitively, however, it is not. It follows that *many* and *nearly all* do not constitute a modal logic, in the sense we have outlined.

It follows that dual negation, by itself, does not provide us with the link between modality and quantifiers in general that it provides between modality and the classical quantifiers. We can establish a very close relationship of a different sort, however, by extending the treatment we gave to negation to the binary truth-functional operators. This will enable us to generalize the mode of lexical representation that we developed for quantifiers in Chapter 1 to include modal operators, such as adverbs, as well.

Section 2: Conjunction and Disjunction of Quantifiers

Definition 13 can be extended to binary truth-functional operators as follows:

Definition 15 (Outer Connection): Let Q_1 and Q_2 be quantifiers such that there are elementary quantifiers q_1 in Q_1 and q_2 in Q_2 such that

$$(117) R_{q_1}^{n1} = (n_1; R_1^1, R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k)$$

$$(118) R_{q_2}^{n2} = (n_2; R_1^2, R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k).$$

Let c be a truth-functional mapping from R^2 into R . A quantifier Q is said to be the outer connection c of Q_1 and Q_2 or the outer c of Q_1 and Q_2 , written

$$(119) Q = c(Q_1, Q_2) \quad \text{or} \quad Q = Q_1 c Q_2$$

if there is an elementary quantifier q in Q such that

$$(120) R_q^n = (n; c(R_1^1, R_1^2), R_2, R_3, \dots, R_{k-2}, R_{k-1}, R_k)$$

where $n = \max(n_1, n_2)$. For any c , $c(Q_1, Q_2)$ is said to be an outer connection of Q_1 and Q_2 .

We could also define notions of inner and dual connection, along the lines of Definition 13, as mappings from $R^n \times R^n$ into R , but, although these mappings are worth studying, they are of no use to us here. Much of the discussion in Partee (1970) can be interpreted as an informal account of some of the properties of inner conjunction.

As might be expected, the instances of Definition 15 that will really be of interest to us are those in which c is conjunction or disjunction. One example of outer conjunction that we have seen is Altham's analysis of *a few*. As we saw in 11(32), *a few* can be analyzed in terms of *some* and *many*. Since *many* is of order 2, we will consider the relativized case, which, as we saw in 11(61), can be written as

$$(121) (A \text{ few } \alpha)(\Psi, \phi) = (\text{Some } \alpha)(\Psi, \phi) \wedge -(\text{Many } \alpha)(\Psi, \phi).$$

We know that the quantificational relation expressed by *many* is given by

$$(122) R_{\text{many}} = (2; \geq n, K, n_{123})$$

as we saw in (21), and that the set-theoretic relation expressed by relativized *some* is given by

$$(123) R_{\text{some}}^2 = (2; \perp, n_{12}, \text{Comp}_3)$$

as we saw in (109). According to Definition 13, the quantificational relation expressed by *not many*, since it is the same as $(\text{many})^+$, is given by

$$(124) R_{(\text{many})^+} = (2; < n, K, n_{123})$$

which is also the same as R_{few} , because *few* = *not many*. Formula (121) suggests that a *few* is the outer conjunction of *some* and (*many*)⁺. We can show that this is true by finding set-theoretic relations R , and D , such that (123) can be written as (117), (124) can be written as (118), and the set-theoretic relation expressed by (121) can be written as (120).

To apply Definition 15 we first try to find the set-theoretic relation expressed by relativized a *few* directly from (121). According to (121), the formula

$$(125) \quad (A \text{ few } \alpha)(\Psi; \Phi)$$

is true if and only if both

$$(126) \quad (\text{Some } \alpha)(\Psi; \Phi)$$

and

$$(127) \quad (\text{Many}^+ \alpha)(\Psi; \Phi)$$

are true. Since (126) is true under f if and only if (123) holds and (127) is true if and only if (124) holds, it follows that (125) is true if and only if both (123) and (124) hold. This tells us that the set-theoretic relation expressed by (125) is the conjunction of (123) and (124), which is given by

$$(128) \quad ((\mathbb{E}, n_{12}, \text{Comp}_3) \wedge (<n, K, n_{123})) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>}$$

In other words, f satisfies (125) if and only if (128) holds.

Formula (128) is just another way of writing

$$(129) \quad (\mathbb{E}, n_{12}, \text{Comp}_3) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>}$$

$$\wedge (<n, K, n_{123}) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>}$$

which expands in the usual way to

$$(130) \quad (D \cap \Psi_f^\alpha \cap \Phi_f^\alpha \neq \lambda) \wedge (K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) < n)_{<DR><DR>}$$

In other words, the set-theoretic relation expressed by (125) is given by the conjunction of

$$(131) \quad D \cap \Psi_f^\alpha \cap \Phi_f^\alpha \neq \lambda_{<DR><DR>}$$

and

$$(132) \quad K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) < n_{<DR><DR>}$$

as we might have expected. Unfortunately, it still is not clear how (131) and (132) apply to Definition 15.

This is not difficult to discern, however. Formula (132) says something about the cardinality of the intersection of D , $s^*(\Psi)$, and $s^*(\Phi)$. Formula (131) also says something about this intersection, but not, at least explicitly, about its cardinality. If we can reformulate (131) in terms of cardinality, then we will be well on our way to the outer conjunction, because we will have made (131) and (132) comparable.

First we notice that what (131) says about the intersection of the three sets is that it is not empty. This, however, is just another way of saying that its cardinality is greater than zero, so we can reformulate (131) as

$$(133) \quad K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) > 0_{<DR><DR>}$$

Given (133) we can rewrite (130) as

$$(134) \quad (K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) > 0) \wedge (K(D \cap \Psi_f^\alpha \cap \Phi_f^\alpha) < n)_{<DR><DR>}$$

and, since the explicit form of the relation expressed by (133) is

$$(135) \quad (2; >0, K, n_{123})$$

we can rewrite (129) as

$$(136) \quad (>0, K, n_{123}) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>} \wedge (<n, K, n_{123}) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>}$$

This enables us to write (128) in the form

$$(137) \quad ((>0, K, n_{123}) \wedge (<n, K, n_{123})) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>}$$

which collapses very naturally into

$$(138) \quad (0 < \dots < n, K, n_{123}) (D, \Psi_f^\alpha, \Phi_f^\alpha)_{<DR><DR>}$$

as the formula for the set-theoretic relation $R_{\text{a few}}$.

Formula (137) says that the cardinality of the intersection is greater than 0 and less than n , while (138) says that it is between 0 and n . These are clearly just two ways of saying the same thing. It follows that

$$(139) R_{a \text{ few}} = (2; 0 < \dots < n, K, n_{123})$$

is the set-theoretic relation expressed by *a few* in its relativized form (121). It is also not difficult to show that (139) is the quantificational relation expressed by *a few*, because *a few* is clearly of order 2.

In terms of Definition 15, if we take

$$n_1 = n_2 = 2$$

$$k = 3$$

$$R_1^1 = (>0)$$

$$R_1^2 = (<n)$$

$$R_2 = K$$

$$R_3 = n_{123}$$

$$c = \Lambda$$

then (117) becomes (123), and its equivalent form (135), (118) becomes (124), and (120) becomes (139). It follows that *a few* is, in fact, the outer conjunction of the two quantifiers *some* and the outer negation of *many*, as we were trying to show.

Section 3: Outer Connection and the Classical Negation Set

The most interesting aspect of outer connection, for our present purposes, is the interaction of outer conjunction and disjunction with the negation set of the classical quantifiers. In general, the outer connection of a quantifier and each member of its negation set form a three-way partition, as follows:

Definition 16 (Constant, Redundant, and Significant

Outer Connections): Let Q_1 and Q_2 be quantifiers, c a truth-functional mapping from R^2 into R , and n the order of $c(Q_1, Q_2)$. The outer connection $c(Q_1, Q_2)$ is said to be

(a) constant, if

$$R_{c(Q_1, Q_2)}(D, s^*(\phi_1), s^*(\phi_2), \dots, s^*(\phi_n))$$

is either logically true for all $\phi_1, \phi_2, \dots, \phi_n$ or logically false for all $\phi_1, \phi_2, \dots, \phi_n$;

(b) redundant, if

$$c(Q_1, Q_2) = Q_i \text{ for either } i=1 \text{ or } i=2;$$

(c) significant, if it is neither constant nor redundant.

We have already seen that the outer conjunction of *some* and *few* is significant, because it coincides with *a few*, which is a different quantifier from either *some* or *few*, and it produces neither only logically true nor only logically false truth conditions.

If we let Q_1 and Q_2 vary over the union of the unit set and negation set of a specific quantifier Q , then sixteen outer connections result for any given c , as follows:

$$1. Q \text{ c } Q$$

$$2. Q \text{ c } Q'$$

$$3. Q \text{ c } Q^+$$

$$4. Q \text{ c } Q^*$$

$$5. Q' \text{ c } Q$$

$$6. Q' \text{ c } Q'$$

$$7. Q' \text{ c } Q^+$$

$$8. Q' \text{ c } Q^*$$

$$9. Q^+ \text{ c } Q$$

$$10. Q^+ \text{ c } Q'$$

$$11. Q^+ \text{ c } Q^+$$

$$12. Q^+ \text{ c } Q^*$$

$$13. Q^* \text{ c } Q$$

$$14. Q^* \text{ c } Q'$$

$$15. Q^* \text{ c } Q^+$$

$$16. Q^* \text{ c } Q^*.$$

If we take c to be conjunction or disjunction, then we can ignore 5, 9, 10, 13, 14, and 15 because of the commutativity of c . These are repeated as 2, 3, 7, 4, 8, and 12, respectively. Numbers 3 and 8 are constant because of the laws of excluded middle and non-contradiction. If c is conjunction, then they are both logically false, as in Definition 16(a), and if c is disjunction, then they are both logically true. We can also see immediately that 1, 6, 11, and 16 are redundant, because they are outer conjunctions or disjunctions of a single quantifier with itself. It follows that 2, 4, 7, and 15 are the only possibly significant outer conjunctions or disjunctions on the list.

For c =conjunction we get the (a) members of the following pairs and for c =disjunction we get the (b) members:

(140) (a) $Q \wedge Q'$ (b) $Q \vee Q'$

(141) (a) $Q \wedge Q^*$ (b) $Q \vee Q^*$

(142) (a) $Q' \wedge Q^+$ (b) $Q' \vee Q^+$

(143) (a) $Q^* \wedge Q^+$ (b) $Q^* \vee Q^+$

as the significant outer conjunctions and disjunctions of a quantifier Q and the members of its negation set. If we let Q =all, then both (a) and (b) of (140) are redundant, because (a) reduces to all and (b) reduces to all', which is some. In (141), (a) is identically false and thus constant, because all and all* cannot both be true, and (b) is significant, because all \vee all* is a quantifier different from both all and all* that produces both true and false sentences from propositional functions it operates on. Quantifier (142a) is also significant for a similar reason, but (142b) is constant, because its truth condition is identically true. Both (a) and (b) of (143) are redundant, because (a) reduces to all* and (b) reduces to all+.

It follows that

(144) all \vee all*

(145) all' \wedge all+

are the only significant outer conjunctions or disjunctions of all and its three negations. Examples of (144) include sentences like

(146) *Everyone or no one now understands quantifiers.*

which is represented semantically by

(All \vee All*x) Understands-quantifiers(x)

because all*=no. Examples of (145) include sentences like

(147) *Someone but not everyone now understands quantifiers.*

which is represented semantically by

(All' \wedge All+ x) Understands-quantifiers(x)

because all' = some.

If we let Q =some, then (140a) is redundant, because it reduces to some', and (140b) is redundant, because it reduces to some. Quantifier (141a) is significant, but (141b) is identically true and so constant. In (142), (a) is constant, because it is identically false, and (b) is significant. Both (a) and (b) of (143) are redundant, because (a) reduces to some+ and (b) reduces to some*. This gives us

(148) some \wedge some*

(149) some' \vee some+

as the only significant outer conjunctions or disjunctions of some and its three negations. Examples of (148) include sentences like

(150) *Someone now understands quantifiers and someone doesn't.*

which is represented semantically by

(Some \wedge Some* x) Understands-quantifiers(x)

because some* = some not. Examples of (149) includes sentences like (146), because some' = all and some+ = all* = no. Not surprisingly, (150) is logically equivalent to (147), because some = all' and some* = all+.

The functions that produce (144), (145), (148), and (149) do not form a group, as we saw the negation functions do in the Theorem of Chapter 3, but, together with the negation functions, they do give us the following structural result:

Theorem: Let h_i , $i=1,2,3$, be functions defined on the class of quantifiers as follows: For all quantifiers Q ,

$$h_1(Q) = Q^+ \wedge Q'$$

$$h_2(Q) = Q \vee Q^*$$

$$h_3(Q) = Q^+ \vee Q'$$

$$h_4(Q) = Q \wedge Q^*$$

Let g_i , $i=1,2,3$, be the negation functions and g_4 the identity function on the class of quantifiers. The following relation holds between the g_i and the h_j :

$$(151) \quad g_i \circ h_j = g_{i+2} \circ h_j = h_i \circ h_j = h_{i+2} \circ h_j = \begin{cases} h_{j+1} & \text{if } j \text{ is odd} \\ h_{j-1} & \text{if } j \text{ is even} \end{cases} \quad \text{if } i=1$$

$$h_j \quad \text{if } i=2$$

Proof: Define $Q^! = g_4(Q) = Q$. The relations

$$h_1^! = h_1^+ = h_2$$

$$h_2^! = h_2^+ = h_1$$

$$h_3^! = h_3^+ = h_4$$

$$h_4^! = h_4^+ = h_3$$

and

$$h_1^* = h_1^! = h_1$$

$$h_2^* = h_2^! = h_2$$

$$h_3^* = h_3^! = h_3$$

$$h_4^* = h_4^! = h_4$$

interpreted in the obvious way, are easily verified and collapse naturally into (151).

Q.E.D.

Formula (151) is of no further use to us and is included only for the sake of completeness, because it summarizes the relationship between the negation functions g_i and the *all-some* significant outer connection functions h_j . The h_j themselves are of

considerable further importance, however, because it is they that provide the link between quantification and modality that we are trying to establish, as we will see in the next section.

Section 4: Quantifiers and Modalities

4.1: Modalities and State-Descriptions

The results of the preceding section can be summarized as in Table 1. The first column of the table

function of Q	Q = all	Q' = some	function of Q'
$g_4(Q) = Q$	Q	-Q'	$g_1(Q') = Q$
$g_2(Q) = Q^*$	Q-	-Q'	$g_3(Q') = Q^*$
$h_1(Q) = Q^+ \wedge Q'$	-Q \wedge -Q-	Q' \wedge -AQ'	$h_4(Q') = Q^+ \wedge Q'$
$g_3(Q) = Q^+$	-Q	Q'-	$g_2(Q') = Q^+$
$g_1(Q) = Q'$	-Q-	Q'	$g_4(Q') = Q'$
$h_2(Q) = Q \vee Q^*$	Q \vee Q-	-Q'- \vee -Q'	$h_3(Q') = Q \vee Q^*$

Table 1

contains the identity and negation functions and the *all-some* significant outer conjunction and disjunction functions for the unit plus negation set of *all*. The second column contains the expansions of these functions in terms of $Q=all$. In the fourth column we have the same functions for *some* ($=all'$) expressed in terms of *all* and in the third column we have the expansions of these functions in terms of $Q=some$. Specifying that $Q=all$ in the second column and that $Q'=some$ in the third column is important, because the outer connection functions h_j may not be significant for other choices of Q .

The reason for listing the entries in Table 1 in the order given becomes clear, when we compare the table with the following table from Carnap (1956, p. 175):

Modal Property of a Proposition	With 'N'	With '◇'	Semantical Property of a Sentence
Necessary	Np	-◇-p	L-true
Impossible	N-p	-◇p	L-false
Contingent	-Np ∧ -N-p	◇-p ∧ ◇p	Factual
Non-necessary	-Np	◇-p	Non-L-true
Possible	-N-p	◇ p	Non-L-false
Noncontingent	NP V N-p	-◇-p V -◇p	L-determinate

Table 2

Carnap uses "N" in this table to denote logical necessity and "◇" to denote logical possibility. The symbol "p" is a proposition variable. With a little examination, we see that Table 1 and Table 2 are exactly the same, except for the notation. If we replace "N" with "Q" and "◇" with "Q'" and we drop the general argument "p" altogether, then the second and third columns of Table 2 become the second and third columns of Table 1.

The correspondence between these tables suggests that we can treat modalities as quantifiers in a generative grammar. Carnap's own analysis of Table 2 suggests a way of doing this. Carnap begins with the notion of a state-description in a semantical system S_1 , which he defines as "a class of sentences in S_1 which contains for every atomic sentence either this sentence or its negation but not both, and no other sentences" (p.9). By an atomic sentence he means, as usual, "a sentence consisting of a predicate of degree n followed by n individual constants" (p. 5).

Carnap points out that such a set of sentences

obviously gives a complete description of a possible state of the universe of individuals with respect to all properties and relations expressed by predicates of the system. Thus the state-descriptions represent Leibniz' possible worlds or Wittgenstein's possible states of affairs.

(p. 9)

He then defines the following semantical properties of sentences in terms of the satisfaction relation between sentences and the class of state-descriptions:

- (152) A sentence ζ_1 is L-true (in S_1) =_{Df} ζ_1 holds in every state-description (in S_1). (p. 10)
 ζ_1 is L-false (in S_1) =_{Df} $\sim\zeta_1$ is L-true. (p. 11)
 ζ_1 is L-determinate (in S_1) =_{Df} ζ_1 is either L-true or L-false. (p. 11)
 ζ_1 is L-indeterminate or factual (in S_1) =_{Df} ζ_1 is not L-determinate. (p. 12)

and he uses these notions to explicate the various modalities.

The semantical notion corresponding to each modality is displayed in the fourth column of Table 2. By comparing each modality with its semantical explication, we can easily determine the relationship of the modalities to quantification. As we see from the first line of Table 2, Carnap equates necessity with L-truth. According to (152), this means that a proposition is necessary, or necessarily true, if the sentence that expresses it holds in every state-description. This gives us a correspondence between all (=every, for our purposes) and necessity, as suggested by the first lines of our tables. Carnap equates impossibility with L-falsity, so a proposition is impossible, according to (152), if the sentence that expresses its negation is L-true, that is, if this negation sentence holds in every state-description. This gives us the correspondence between impossibility and all* (=all not) that is suggested by the second lines of our tables. Contingency Carnap equates with factuality, so a proposition is contingent, according to (152), if the sentence that expresses it is not L-determinate, that is, neither L-true nor L-false. This means that neither the sentence nor its negation holds in every state-description, giving us the correspondence between contingency and all* ∧ all' that is suggested by the third lines of our tables.

Non-necessity is correlated with non-L-truth in Table 2, so a proposition is non-necessary, if the sentence that expresses it does not hold in every state-description. The correspondence between non-necessity and all* (=not all) that is suggested by the fourth lines of our tables follows. On the fifth line of Table 2 we have possibility equated with non-L-falsity, telling us that a proposition is possible, if the negation of the sentence that expresses it does not hold in every state-description. This gives us the correspondence between possibility and all' that is suggested by the fifth lines of our tables. Finally, Carnap

equates the modality of non-contingency with the property of being L-determinate, so a proposition is noncontingent, if neither the sentence that expresses it nor the negation of that sentence holds in every state-description. This gives us a correspondence between noncontingency and $h_2(all)$, as suggested by the last lines of our tables.

4.2: Lexical Representation of Modalities

The point of Section 4.1 is that each modality that Carnap considers can be expressed in terms of *all* and one of the g_i or h_i functions, as indicated in the second columns of our tables, if we take the quantification to be over the class of state-descriptions. A similar analysis of the third columns of the tables shows that they can be expressed in terms of *some* as well. This gives us a very natural mode of lexical representation for modalities, whether they appear as modal adverbs, verbal auxiliaries, or anything else. All we have to do is to introduce a fifth value for the universal domain variable "D", in addition to the four that we allowed in IV,2,1. Just as D_1 is the set of things, D_2 is the set of people, D_3 is the set of times, and D_4 is the set of places, so D_5 will be the set of state-descriptions or "possible worlds".

The notion of "possible worlds" has figured prominently in modern logic for some time and has more recently been applied to problems of linguistic theory. Kripke (1963) discusses a much more sophisticated formalization than Carnap's notion of state-description and applies it to problems of modal-logic semantics. Lakoff (1968) applies the "possible-worlds" notion to the problem of coreference in natural language and Cushing (1972b) investigates its possible relevance to the phenomenon of sentence pronominalization. As we now see, it also provides us, in conjunction with the formal semantic theory of quantification that we have developed here, with a natural and revealing way of handling modalities in natural language in the semantic component of a generative grammar.

If we add D_5 to the restricted set of universal domains recognized by our semantic theory, then modal adverbs like *necesessarily* and *possibly* can be represented semantically in the lexicon as

necesessarily: $all, D=D_5$

possibly: $g_1(all), D=D_5$

respectively. *All* itself, of course, will be represented in terms of its explicit quantificational relation, as we saw in IV,1. A sentence like

Everyone may now understand quantifiers.

which is normally interpreted contingently, can be represented semantically as the simple quantification (occurrence)

(153) $(h_1(all) x_1 (all x_2) (x_1 \in D_5, x_2 \in D_2; x_2 \text{ understands } \dots \text{ quantifiers in } x_1))$

because contingency and *everyone* will be represented as

contingency: $h_1(all), D=D_5$

everyone: $all, D=D_2$

respectively. Definition 11 deals explicitly only with semantic representations of simple occurrences that contain only one instance of " $x \in D$ ", but it is easily generalized to account for formulas like (153) that contain two or more such instances.

We can also use this mode of lexical representation for modalities other than those which Carnap considers. This constitutes empirical support for our framework in the form of predictive power for data not yet analyzed. All we have to do, in fact, to account for other modalities is to choose quantifiers other than *all* (or *some*). Modal adverbs like *probably* and *unlikely*, for example, can be represented semantically in the lexicon as

probably: $most, D=D_5$

unlikely: $g_3(most), D=D_5$

respectively, and sentences like

Probably someone still doesn't understand quantifiers.

can be represented semantically in the form

$(Most x_1) (g_2(Some) x_2) (x_1 \in D_5, x_2 \in D_2; x_2 \text{ understands } \dots \text{ quantifiers in } x_1))$

A modal adverb like *most certainly* can be represented lexically as

almost certainly: $g_1(\text{many})$, $D=D_5$

because *nearly all*=(*many*)', and so on. The fact that our mode of lexical representation can be used for a class of data more general than the one that was used in its construction reflects well on both the correctness and the explanatory power of the theory that underlies it.

Section 5: Truth, Existence, and the Bar Notation

As Joseph Emonds (personal communication) has pointed out to me, an important point remains to be made before we conclude our study. Cushing (1972b) argues that there is good reason to expect that sentences and noun phrases will have parallel semantic structures, as follows:

In projective geometry there is a systematic parallelism between points and lines. There is a 'principle of duality' which states that for any theorem about points there is a parallel theorem about lines, and vice versa. I do not know whether such a strong principle can be shown to hold for general formal systems, but that there is considerable parallelism of a weaker sort is clear. A formal language contains terms and (well-formed) formulas. The terms are taken to refer to objects in some model and the formulas are taken to refer to situations involving objects in the model. There is thus a parallelism between the existence of an object-referent for a term and the existence of a situation-referent for a formula, i.e., between the existence of a referent for a term and the truth-value of a formula (that is, of the proposition represented by the formula). We can conclude from this that terms and formulas have, in this sense, parallel semantic structures. In natural languages the role of term is played by the theoretical construct NP and the role of formula is played by the theoretical construct S. We are thus led to expect NP's and S's to have, in a real sense, parallel semantic structures.

(pp. 191-192)

As an example of this parallelism, Cushing examines the phenomenon of sentence pronominalization in English and compares it to the more familiar process of noun-phrase pronominalization. He points out that

there is a feature [+definite] on NP's in English that determines what pronoun an NP can be coreferential with. A [+definite] NP can be coreferential with *it* and a [-definite] NP can be coreferential with *one* if positive and *none* if negative. We get

The Hatter ate a piece of the cake and Alice ate a piece of it too.

The Hatter ate a piece of the cake and Alice ate one too.

The Hatter ate a piece of the cake, but Alice ate none.

where underlining indicates coreferentiality.

(p. 192)

Having already shown that "there is a feature [+F] on S's in English that determines what pronoun an S can be coreferential with", so that "a [+F] S can be coreferential with *it*, just as a [+definite] NP can, and a [-F] S can be coreferential with *so* if positive and *not* if negative" (p. 192), Cushing goes on to argue that the NP feature [definite] and the sentence feature [F] are really one and the same, so that "both NP's and S's are specified with respect to the feature [+definite]...." (p. 192).

Cushing points out that

These conclusions bear out the suggestion by the Kiparskys (1968) that perhaps 'there is a syntactic and semantic correspondence between truth and existence ...at some sufficiently abstract level of semantics', though in a somewhat different form from what they expected.

(p. 193)

and then goes on to consider the syntactic aspect of this correspondence. First he points out that

Arguing entirely from the syntax of English, independently of logical and semantic considerations, Chomsky (1970) and Jackendoff (1968b) have shown that NP's and S's have parallel syntactic structures, as well as the parallel semantic structures I have argued for here. To capture this syntactic parallelism formally, Chomsky-Jackendoff proposed the following formal notations:

\overline{N} = N+complement

\overline{V} = VP

\overline{N} = NP

\overline{V} = S

and the convention that parallel facts about, or processes occurring in, NP's and S's should be stated in terms of X.

(p. 193)

He then argues that "This bar notation, plus the fact that [+F] is really [+definite], enables us to express both NP and S pronominalization as a single unified process" (p. 193).

The specific analysis and rules that Cushing develops for NP and S pronominalization need not concern us here. The important point, from our present point of view, is that a significant parallelism was shown to exist on both the syntactic and semantic levels between sentences and noun phrases. Two seemingly unrelated processes were shown to be instances of a single underlying process, manifesting itself in one way, when it occurs in NP's, and in another, when it occurs in S's.

This is exactly what we found in the case of quantifiers and modalities, as far as their semantics is concerned, in the first four sections of this chapter. It is not difficult to see, in fact, that the same conclusion can be drawn for time and place adverbials as well, based on our discussion in IV,2,1. There seems, on the surface, to be a significant semantic difference between the forms *something*, *someone* in

(154) (a) *Something is interesting.*

(b) *Someone finds linguistics interesting.*

on the one hand, and the forms *sometimes*, *somewhere*, *may* (or *possibly*) in

(155) (a) *Linguistics is sometimes interesting.*

(b) *Somewhere linguistics is found interesting.*

(c) *Linguistics may (possibly) be the most interesting of all.*

on the other. *Something* and *someone* in (154) are noun phrases acting as the subjects of their respective sentences. They are clearly related to the instances of *some* in

(156) (a) *Some things are interesting.*

(b) *Some people find linguistics interesting.*

respectively, which act as modifiers of the nouns *things* and *people*. *Sometimes*, *somewhere*, and *may* in (155), however, are a very different kind of form. Rather than acting as the modifier of a noun phrase, each of these forms acts as a modifier of its entire sentence. *Sometimes* in (155a) tells us that the sentence

(157) (a) *Linguistics is interesting.*

is true at some times. *Somewhere* in (155b) tells us that the sentence

(157) (b) *Linguistics is found interesting.*

is true at some place. *May* in (155c) tells us that the sentence

(157) (c) *Linguistics is the most interesting of all.*

may, *possibly*, be true. The *some* forms in (154) and (156) are all more or less nominal in character, functioning either as noun phrases or as modifiers of noun phrases, but the relevant forms in (155) are all, loosely speaking, modifiers of sentences; a time adverbial, a place adverbial, and a part of the verbal auxiliary.

Despite this striking surface semantic difference, however, we have found, in IV,2,1 and the last four sections, that the relevant forms in (154) and (155) are really exactly the same on a deeper level. The sentences in (154) can be represented semantically in the form

(158) (a) $(g_1(\text{all}) x)(x \in D_1; \text{Interesting}(x))$

(b) $(g_1(\text{all}) x)(x \in D_2; \text{Finds-interesting}(x, \text{Linguistics}))$

respectively, and the sentences in (155) can be represented semantically in the form

(159) (a) $(g_1(\text{all}) x)(x \in D_3; \text{Interesting}(\text{linguistics}) \text{ at } x)$

$(g_1(\text{all}) x)(x \in D_4; \text{Found-interesting}(\text{linguistics}) \text{ at } x)$

$(g_1(\text{all}) x)(x \in D_5; \text{Most-interesting}(\text{linguistics}) \text{ in } x)$

respectively. Although there are differences, of course, in the quantified predicates of these formulas, which reflect other aspects of the meanings of the sentences they represent, we see that the forms we are presently interested in are all represented in exactly the same way, as the dual negation of *all*. The difference in meaning arises solely from the different choices of D. *Something*, *someone*, *sometimes*, *somewhere*, and *may* can be listed in the lexicon, respectively, as

something: $g_1(\text{all}), D=D_1$

someone: $g_1(\text{all}), D=D_2$

sometimes: $g_1(\text{all}), D=D_3$

somewhere: $g_1(\text{all}), D=D_4$

may: $g_1(\text{all}), D=D_5$

as we saw in the last section. Although the first two of these forms are NP forms in surface meaning and the last three are S forms, our theory has revealed that they are really fundamentally the same in their underlying semantics.

Chomsky (1970) shows, in effect, that there is also a syntactic parallelism between quantifiers and modalities to go along with the semantic parallelism that we have developed here. As a part of the bar notation that we saw earlier, he introduces the notion of "specifier" with the rule

$$\bar{X} \rightarrow [\text{Spec}, \bar{X}] \bar{X}$$

"where [Spec, \bar{X}] will be analyzed as the determiner and [Spec, \bar{V}] as the auxiliary (perhaps with time adverbials associated)" (p. 210) and he demonstrates a substantial parallelism in the syntactic behavior of [Spec, \bar{X}] for $X=N$ and $X=V$. Since quantifiers occur syntactically in English as part of the determiner and modals occur as part of "the verbal auxiliary (perhaps with time adverbials associated)", this syntactic result corresponds very nicely with the semantic analysis that we have developed.

None of these results should be surprising, if we look again at the quote with which we began this section:

There is a parallelism between the existence of an object-referent for a term and the existence of a situation-referent for a formula, i.e., between the existence of a referent for a term and the truth-value of a formula.

As we have noted again and again, a quantifier is a semantic operator that answers one of the questions *How many?* or *How much?* In view of the fact that a quantifier in a noun phrase tells us *How many?* object-referents the NP has, this parallelism should lead us to look for quantifiers that tell us *How many?* situation-referents an S has. As we have seen this role is played by what turns out on the surface to be modality. Just as taking $D=D_2$ tells us *How many?* people there are who have the property specified by some surface NP, so taking $D=D_5$ tells us *How many?* ways the world could be arranged in which the proposition specified by a surface S would be true, that is, *How many?* possible worlds are correctly described, in part, by S. We see once again, with Cushing (1972b) and the Kiparskys, that "there is a syntactic and semantic correspondence between truth and existence...at some sufficiently abstract level of semantics."

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