

A Note on Order

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The standard way of formally representing generalization is intended to represent three notions simultaneously: force, scope and order. In:

$$\forall x(\phi x),$$

the first of these parameters is indicated by the symbol “ \forall ” standing for universal force; replacing it with the symbol “ \exists ” indicates existential rather than universal force. The second parameter is represented by the bracketing; variation comes to the fore when there is more than one indicator of generalization. Thus:

$$\forall x(\exists y(\psi xy)),$$

differs from:

$$\exists y(\forall x(\psi xy)),$$

in the scope of generalization. The last parameter is indicated by the letter immediately adjacent to the generalization symbol. Variation here is indicated by case. Use of miniscules, indicates that the generalization is of the first-order; use of majuscules, that it is of the second-order. Accordingly:

$$\forall F(\forall x(Fx))$$

contains a second-order generalization, in addition to the first-order generalization.

What the standard notation for generalization is designed to do, and indeed excels at, is displaying propositional structure. It graphically represents how the force, scope and order components interact to make up propositions expressing generalization, and it allows us to typify these propositions in a completely discriminable manner. On a glance, universal generalization can be distinguished from existential, first-order from second-order, first-order universal from second-order existential, and so on for the various combinations of the parameters. The beauty of the notation is that the structure over which both the truth-conditions and inferential capacities of generalizations are defined is represented as a single composition of force, scope and order, and it is this that explains its perseverant utility. It characterizes what we mean by the logical form of generalization.

This representational success is a core part of the analytic story about generalization, but it does beg a foundational question: What is it that makes these representations *logical* representations? The answer is readily at hand: it is because they are made up of logical parts, put together in a logical way. Comprehending this answer, however, requires some prior information - we need to know in what sense force, scope and order are logical notions.

This is, of course, a very big question, but there are expectations we have for any account of this logicity: It should position us to answer a range of fundamental questions about the extents of logical force, scope and order. Are universal and existential the only forces, or are there others? Are the possibilities of scope at a factor of $n!$ for n -many operators, or are there limitations? Or perhaps more possibilities than the $n!$ allowed by linearly ordered operators, as in the case of branching quantifiers? What is the difference between orders of generalization; are there orders greater than two? Are there dependencies among these factors — does scope depend on force, or force on order, for instance?

Frege's epochal breakthrough in understanding generalization was the insight that answers to questions of logicity spring from the alignment of force, scope and order with the semantics, syntax and ontology of generalization. As discussed in Heck and May (2013a), initially in *Begriffsschrift*, Frege develops the core representational aspects of generalization. That Frege focuses on propositional structure is unsurprising, given his emphasis in that work on presenting a notation in which the formality of inference is explicitly represented, in which proofs could only be given in a rigorous, gap-free manner. It is only later, however, when Frege, spurred by pointed criticisms in reviews of *Begriffsschrift*, engages with Boolean logic that he begins to address logicity in the broader context.¹

For Frege, the key is that the fundamental logical notion is that of a function; accordingly, what the logical notation — the conceptual-notation — represents is a structure of function and argument. But if in *Begriffsschrift*, Frege was concerned with justifying the formality of these representations, in *Grundgesetze* he takes this for granted, and shifts his attention to explicating what is represented, the functions themselves. Frege characterizes the functions in terms of their arguments, distinguishing between the base case, in which objects are arguments, and higher cases, in which functions themselves are arguments. The base and higher cases are organized hierarchically: The base case is the first-level; functions that take the base-case as arguments are second-level; those that take second-level functions as arguments are third-level, and so on. Frege's notation directly reflects this hierarchical relation. This, when we write:

$$\Theta(f(x)),$$

what we are representing is a second-level function taking a first-level function as argument. Thus, that the functional ontology is hierarchical fixes both the syntax and the semantics of functional representations; it entails that every well-formed logical formula expresses a well-formed proposition.²

Within this conception, Frege identifies one hierarchy of special importance for logic, the *conceptual* hierarchy. Concepts, by Frege's lights, are functions that have truth and falsity as their values.³ Thus, first-level concepts take objects to truth-values, second-level concepts take first-level concepts to truth-values, third-level concepts take second-level to truth-values, and so on. In this context, Frege analyzes generalization as a second-level concept. Thus, the sentence:

¹This is in 1881, in his paper "Boole's Logical Calculus and the Concept-script". See Heck and May (2013a).

²Of course, the irony is that the most well-known system for which this ideal fails is Frege's own.

³That is, they are characteristic functions. Since for Frege, truth and falsity are logical objects, concepts are mappings from the logical objects to a specified sub-set of the logical objects. See Heck and May (2013b) for discussion.

All men are mortal,

contains the concept

All ζ are ξ ,

which maps a pair of first-level concepts onto a truth-value. Thus, on Frege's view, generalizations express conceptual relations; they are concepts that relate concepts. Their logicity emerges through their definability relative to the functional hierarchy.⁴

Frege's conception of generalization has proved to be extraordinarily fruitful. Notably, we now understand that force distributes over a class of conceptual relations that is inclusive of the generalization relation; these are the generalized quantifiers. From this perspective, universal and existential force are instances of quantificational force, that is, concepts related in terms of the cardinalities of their extensions.⁵ Moreover, Frege's way of characterizing scope has been deeply explored within contemporary syntactic theory.⁶ For the remainder of this brief note, however, we want to offer some remarks on the third aspect, that of the order of generalization, or as we will now consider the matter, the order of quantification.

On the Fregean conception of quantifiers as conceptual relations, the order of quantifiers is directly given by the conceptual hierarchy. Because quantifiers relate concepts, it follows that their lowest order will be second-level relations, that is, relations between basic first-level concepts. The next order up will be third-level relations, relating second-level concepts, and so on up the discrete steps of the hierarchy. Now Frege's way of rendering the orders of quantifiers can be re-phrased. Rather than classifying the quantifiers in terms of their arguments, they can be classified over the types of things that make up the extensions of the arguments. Done this way, we obtain the familiar categorization of quantifiers as *first order*, *second order*, etc, for every finite order in the theory of simple types. Thus, first-order quantifiers range over a domain of objects; second-order quantifiers range over properties of, or relations among, those objects; third-order quantifiers range over properties of properties, relations among properties, and so on. Here we focus on the distinction between first- and second-order quantifiers.

Roughly speaking, the notion of *order* for a quantifier can be articulated either *syntactically* or *semantically*. The syntactical notion of order for a quantifier is determined by the grammatical category that the terms quantified over occupy, whereas the semantical notion is determined by the type-theoretic level (over a given domain of objects) at which the notions being quantified over can be found. Intuitively, we might think of these notions as locked together, given that the role of the syntactic representation is to reflect the semantic characterization. However, when the syntactic and semantic notions are made bit more clear, it turns out that the boundaries between them, as well as the different orders within each, might in fact be more flexible than the intuitive picture would indicate.

⁴Frege's conception bears more than a passing relation to Boole's although there is a fundamental difference in how they see the relation of primary and secondary propositions. See Heck and May (2013a). Note that in Frege's system we can also define non-relational quantifiers, those that take only a single concept as argument.

⁵In this area, our depth of understanding is very much due to Ed Keenan's seminal contributions to the theory of generalized quantifiers. Although Keenan and Stavi (1986) is often singled out (along with Barwise and Cooper (1981) and Higginbotham and May (1981)), this is a sample of Ed's extensive and extremely influential investigations.

⁶See May (1977, 1985).

Consider a quantified statement of the form $Q\alpha\Phi(\alpha)$, where Q is either \exists or \forall and α is a syntactical constituent. We leave the syntactical category of α unspecified to allow for the formula $\Phi(\alpha)$ to be obtained by replacing α for constituents of varying categories. If α replaces a constituent of category NP then the quantifier in $Q\alpha\Phi(\alpha)$, is (syntactically) at the first order; if α replaces a constituent of category VP then the quantifier is at the second-order, etc. There is in fact a tradition going back to Prior (1971) that emphasizes how items of *any* syntactic category — and not just those of category NP — are available for quantification. Prior points out that such “non-nominal” quantifications are ubiquitous in natural language. Not just second-order quantifiers fall under this heading:

He is something I am not — kind;

but also quantifiers binding constituents of other syntactic categories, for instance AdvP:

I hurt him somehow.

Notice that constituents of category AdvP combine with VP’s to return VP’s, and thus can be thought of as representing a mapping from properties into properties, an intrinsically higher-order notion.

From a semantic point of view, quantifiers can be given a treatment that is parallel to the syntactic one. First order quantifiers range over individuals members of the universe of discourse D , so that $\exists x\Phi(x)$ is true if some member of D falls within the extension of Φ , and $\forall x\Phi(x)$ is true if every member of D falls within the extension of Φ . At the second-order, $\exists X\Phi(X)$ is true if some subset of D falls within the extension of Φ , and similarly, $\forall X\Phi(X)$ is true if every subset of D falls within the extension D . From this point of view, (codified in the theory of generalized quantifiers), first-order quantifiers such as \exists and \forall denote collections of subsets of the domain, and $Qx\Phi(x)$ is true if the extension of Φ is among the subsets denoted by Q . And analogously, second-order quantifiers denote collection of *collections* of subsets.

The point to keep in mind is that whether a quantifier is properly characterized, from a semantic point of view, as being first- or second-order is completely determined by the type-theoretic level of the entities it applies to. This is perhaps most clear by considering notions other than quantifiers, such as for instance objectual identity between members of D . There is no question that statements of the form $a = b$ are essentially first-order statements: they hold, or fail to hold, of pairs of (not necessarily distinct) objects. And yet a case can be made that asserting $a = b$ involves higher order notions, in that it implies that every property of a is also a property of b . We propose to express this distinction by saying that the first-order notion of objectual identity *expresses*, but does not *assert*, a second-order claim (see Antonelli and May (2012)).

The same distinction applies in the case of quantifiers, although it might not be as evident unless we broaden our horizon beyond consideration of just the two quantifiers \exists and \forall . In particular it is important to look at *binary* first-order quantifiers, i.e., quantifiers that relate two subsets of the domain (syntactically, these are quantifiers that take not one but two formulas as argument, as mentioned). A prime example is the Aristotelian quantifier All, which relates subsets A and B precisely when $A \subseteq B$. While All clearly involves no reference to notions other than first-order, the case is not as clear for other first-order quantifiers.

Consider for instance the “Frege” quantifier F , relating two subsets A and B precisely when there are no more A ’s than B ’s — or, in Boolos (1981) suggestive rendition, when “for every A there is a (distinct) B .” It is clear that from a semantic point of view, F is a first-order quantifier, just like All : the former, just like the latter, applies to pairs of subsets of D . However, the relation expressed by F involves higher-order notions, since it implies (it is in fact equivalent to) the existence of an injective function mapping the A ’s into the B ’s. We characterize such a distinction by saying that F expresses — but does not assert — the existence of such a function.

The notion of order of a quantifier, whether specified syntactically or semantically, appears therefore not to be fine-grained enough. Quantifiers that are first-order — in that they represent predicates over, or relations between, subsets of the domain — might in fact be quite different to the extent that they involve higher-order notions. That such a finer-grained classification is needed is clear in fact from considering the vastly different expressive power of quantifiers that are, from a semantic point of view, first-order. Consider the Aristotelian quantifier All in comparison to the Frege quantifier F . The latter does not add to the expressive power of ordinary first-order languages, and in turn can be used to express the generalizations \forall and \exists . But the former is vastly more expressive, as discussed in Antonelli (2010). It allows a categorical characterization of the structure \mathbf{N} of the natural numbers, and is therefore not reducible to ordinary first-order logic.

Frege, to his dismay, gave into the seduction of second-order logic, with its invitation to much richer mathematical results than are available in first-order logic. To Quine, second-order logic, with its bloated ontology of *abstracta*, was repugnant. But what our brief remarks highlight is for both Frege and Quine, the concern is with second-order logic, that is, with systems in which second-order claims are asserted. Expression of second-order notions is fundamentally weaker than asserting them, while still allowing for the mathematical richness that gives logicism its grip on our imagination.

References

- Antonelli, G. A. 2010. The nature and purpose of numbers. *The Journal of Philosophy* 107:192–212.
- Antonelli, G. Aldo, and Robert May. 2012. Quantifiers and determiners. In *The routledge companion to the philosophy of language*, ed. Gillian Russell and Delia Graff Fara, 342–54. Routledge.
- Barwise, Jon, and Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4:159–219.
- Boolos, George. 1981. For every A there is a B . *Linguistic Inquiry* 12:465–67.
- Heck, Richard, and Robert May. 2013a. The function is unsaturated. In *The Oxford handbook of the history of analytic philosophy*, ed. Michael Beaney. Oxford University Press. Forthcoming.
- Heck, Richard, and Robert May. 2013b. Truth in Frege. In *The Oxford handbook of truth*, ed. M. Glanzberg. Oxford University Press. Forthcoming.

- Higginbotham, James, and Robert May. 1981. Questions, quantifiers and crossing. *Linguistic Review* 1:41–80.
- Keenan, Ed, and Jonathan Stavi. 1986. A semantic characterization of natural language determiners. *Linguistics and Philosophy* 9:253–326.
- May, Robert. 1977. The grammar of quantification. Doctoral Dissertation, Massachusetts Institute of Technology.
- May, Robert. 1985. *Logical form: Its structure and derivation*. Cambridge, MA: MIT Press.
- Prior, Arthur. 1971. *Objects of thought*. Oxford at the Clarendon Press. ix–175 pp. (P. T. Geach and A. J. P. Kenny, eds).

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